OCU-PHYS 284
ITEP/TH-58/07

Boundary Ring or a Way to Construct Approximate NG Solutions with Polygon Boundary Conditions II. Polygons $\bar{\Pi}$, which admit an inscribed circle

H.Itoyama and A.Morozov

Osaka City University, Japan
ITEP, Moscow, Russia

ABSTRACT

We further develop the formalism of arXiv:0712.0159 for approximate solution of Nambu-Goto (NG) equations with polygon conditions in AdS backgrounds, needed in modern studies of the string/gauge duality. Inscribed circle condition is preserved, which leaves only one unknown function $y_0(y_1, y_2)$ to solve for, what considerably simplifies our presentation. The problem is to find a delicate balance – if not exact match – between two different structures: NG equation – a non-linear deformation of Laplace equation with solutions non-linearly deviating from holomorphic functions, – and the boundary ring, associated with polygons made from null segments in Minkovski space. We provide more details about the theory of these structures and suggest an extended class of functions to be used at the next stage of Alday-Maldacena program: evaluation of regularized NG actions.

Contents

1 Introduction

In this paper we begin consideration of the next class of approximate solutions to Nambu-Goto (NG) equations with null-polygon boundary conditions by the method suggested in [1]. This problem is important for the study of the string/gauge (AdS/CFT) duality [2, 3], reformulated recently [4]-[28] as an identity between regularized minimal areas in AdS_5 and BDS/DHKS/BHT [29, 7, 8, 17] amplitudes for gluon scattering in $N = 4$ SUSY YM. Unfortunately, even after this ground-breaking reformulation [4], explicit check of duality is escaping, even in the leading order of the strong-coupling expansion – as usual because of the technical difficulties on the stringy side. In this particular case the first hard problem is explicit solution to a special version of Plateau minimal-surface problem [30]: to Nambu-Goto equations in AdS_5 geometry with the boundary conditions at the AdS boundary, represented by a polygon Π with n light-like (null) segments. We refer to [4] for explanation of how this polygon emerges in the problem after a sequence of transformations,

$$\text{NG model} \rightarrow \sigma-\text{model} \quad \stackrel{T-\text{duality } a\ la\ [31]}{\longrightarrow} \quad \sigma-\text{model} \rightarrow \text{NG model},$$

and to [13, 23] for additional comments and notations. Irrespective of these motivations, the current formulation of the gauge/string duality is now made pure geometric, at least in the leading order:

$$\text{regularized}\Big(\text{area of a minimal surface in } AdS_5,\ \text{bounded by } \Pi\Big) = \text{regularized}\left(\oint_\Pi\oint_\Pi \frac{dy^\mu dy'_\mu}{(y-y')^2}\right) \tag{1.1}$$

and the **first problem** is to find what the minimal surface is (with problems of regularization and higher-order corrections arising at the next stage). As surveyed in [1], explicit solution to the **first problem** is currently available only for $n \leq 4$ [32, 4] and the maximally symmetric case ($\Pi = S^1$) at $n = \infty$. As usual with Plateau problem, even approximate methods are not immediately available beyond this exactly-solvable sector. In [1] an line-of-attack was suggested and the first approximate results obtained for the simplest Z_n configurations. The present paper describes the next step in the same direction, generalizing the results of [1] to the next non-trivial case: of polygons which do not have symmetry, but still have a restricted geometry, identified as "$\bar{\Pi}$ possesses an inscribing circle" in [1]. In this case the boundary conditions and thus the solution are lying in AdS_3 subspace of AdS_5, and the problem is reduced to finding a single non-trivial function, say $y_0(y_1, y_2)$, while the other two are expressed through the AdS_3 constraints [4],

$$\begin{array}{cc} y_3 = 0 & \qquad (Y_3 = 0) \\ y_0^2 + 1 = y_1^2 + y_2^2 + r^2 & \qquad (Y_4 = 0) \end{array} \tag{1.2}$$

2 Approach to approximate solution: a summary of [1]

The strategy, suggested in [1] was to:

• First, represent y_0 as a power series,

$$y_0 = \sum_{i,j\geq 0} a_{ij} y_1^i y_2^j \tag{2.1}$$

and rewrite NG equations in the form of recurrence relations for a_{ij}, with recursion relating the two adjacent "levels" $k = i + j$ and leaving a number of free parameters. If the structure of the *boundary ring* is explicitly known, then expansion (2.1) can be modified in order to take boundary conditions into account from the very beginning, though this can cause additional convergency problems for the series.

• Second, truncate the series at some level N and specify the remaining free parameters which match the boundary conditions in the best possible way at given truncation level. Increasing N provides better and better fit to both the NG equations and boundary conditions.

• Fitting criteria and thus the resulting approximations can be different, depending on the further application. As explained in [1], one can improve either local or global approximation to boundary conditions or instead try to better match the behavior at the angles of the polygon, which is responsible for the main IR divergence of the regularized area of the minimal surface.

2.1 Recurrent relations and free parameters

The first recurrence relations were already found in [1]:

There are no relations at levels zero and two: all the corresponding coefficients, a_{00} and a_{10}, a_{01} are free parameters, i.e. there are $\nu_0 = 1$ and $\nu_1 = 0$ of them.

At level two NG equations impose a single relation:[1]

$$a_{11} = -\frac{a_{02}(1+a_{00}^2-a_{10}^2)+a_{20}(1+a_{00}^2-a_{01}^2)}{a_{01}a_{10}} = -\frac{a_{02}A_{10}+a_{20}A_{01}}{a_{01}a_{10}} \tag{2.2}$$

where $A_{01} = 1+a_{00}^2-a_{01}^2$ and $A_{10} = 1+a_{00}^2-a_{10}^2$. Next formulas involve a generalization of these quantities:

$$A_{kl} = 1+a_{00}^2-ka_{10}^2-la_{01}^2 \tag{2.3}$$

At level three we get two relations:

$$a_{12} = \frac{6a_{03}a_{01}^3a_{10}^2A_{10} - 3a_{30}a_{01}^2a_{10}A_{01}^2 - a_{01}^2a_{10}^2(a_{11}^2-4a_{02}a_{20})A_{03} + a_{00}a_{01}^2\big(a_{02}(A_{10}A_{21}+4a_{01}^2a_{10}^2)+a_{20}A_{01}A_{41}\big)}{a_{01}^2a_{10}(A_{01}A_{10}-4a_{01}^2a_{10}^2)},$$

$$a_{21} = \frac{6a_{30}a_{10}^3a_{01}^2A_{01} - 3a_{03}a_{10}^2a_{01}A_{10}^2 - a_{01}^2a_{10}^2(a_{11}^2-4a_{02}a_{20})A_{30} + a_{00}a_{10}^2\big(a_{20}(A_{01}A_{12}+4a_{01}^2a_{10}^2)+a_{02}A_{10}A_{14}\big)}{a_{01}a_{10}^2(A_{01}A_{10}-4a_{01}^2a_{10}^2)}$$

and $\nu_3 = 2$ free parameters a_{03} and a_{30}.

Similarly, at level k there will be $k-1$ relations imposed by NG equations, and $\nu_k = 2$ out of $k+1$ coefficients $a_{i,k-i}$ at this level will remain free. We always choose a_{k0} and a_{0k} for these free parameters. They can be associated with two arbitrary functions – of y_1 and y_2 respectively, and this freedom resembles the general solution of the archetypical equation $\frac{\partial^2 Y}{\partial y_1 \partial y_2} = 0$, given by $Y(y_1,y_2) = f(y_1)+g(y_2)$ with two arbitrary functions f and g. We shall see in s.3 below that even more relevant can be analogy with the ordinary Laplace equation, solved by arbitrary holomorphic and antiholomorphic functions.

If series (2.1) is truncated at level N, it contains $(N+1)^2$ different coefficients a_{ij}, of which $2N+1$ remain free parameters, unconstrained by NG equations.

2.2 Boundary conditions and the boundary ring

According to [1] the boundary conditions can be formulated in terms of the boundary ring $\mathcal{R}_\Pi$, which consists of all polynomials of y-variables that vanish on the boundary polygon Π.[2] Since $y_3 = 0$ this $\mathcal{R}_\Pi$ includes y_3 as a generator and we can actually restrict considerations to polynomials, depending on only three variables y_0, y_1, y_2.

As further explained in [1], if n edges of Π are defined by the equations:

$$\left\{ \begin{array}{c} c_a y_1 + s_a y_2 = h_a, \\ y_0 = y_{0a} + \sigma_a(-s_a y_1 + c_a y_2), \end{array} \right. \tag{2.4}$$

with $c_a = \cos\phi_a$, $s_a = \sin\phi_a$ and $\sigma_a = \pm 1$, see Fig.1, then

• the condition that Π closes in y_0 direction is

$$\sum_{a=1}^{n} \sigma_a l_a = 0, \tag{2.5}$$

where l_a are the lengths of $\bar{\Pi}$, which is projection of Π onto the (y_1,y_2) plane, and

• the following three polynomials are the obvious elements of $\mathcal{R}_\Pi$:

$$\begin{aligned} P_\Pi(y_1,y_2) &= \prod_{a=1}^n \Big(c_a y_1 + s_a y_2 - h_a\Big), \\ \tilde{P}_\Pi(y_0,y_1) &= \prod_{a=1}^n \Big(y_1 + (-)^{a+1} s_a (y_0 - y_{0a}) - c_a h_a\Big), \\ \tilde{\tilde{P}}_\Pi(y_0,y_2) &= \prod_{a=1}^n \Big(y_2 + (-)^a c_a (y_0 - y_{0a}) - s_a h_a\Big) \end{aligned} \tag{2.6}$$

[1] As clear already from this formula the choice of a_{k0} and a_{0k} (instead of, say, a_{k0} and $a_{k,k-1}$) makes the limit $a_{10}, a_{01} \to 0$ singular. Note that original solution of [4] is exactly of this kind: $a_{ij} = \delta_{i1}\delta_{j1}$ and singularities are easily resolved for it.

[2] By definition, solutions of our problem belong to the intersection of the space of $r = 0$ asymptotes of NG solutions with the completion of the boundary ring. In still other words, anzatze that we substitute into NG equations should be taken from completion of the boundary ring. Completion here means first, that, say, r^2 rather than r itself belongs to $\mathcal{R}_\Pi$ according to (1.2), i.e. r belongs to the *algebraic* completion of the ring. Second, our anzatze can be looked for among formal series made out of elements of $\mathcal{R}_\Pi$.

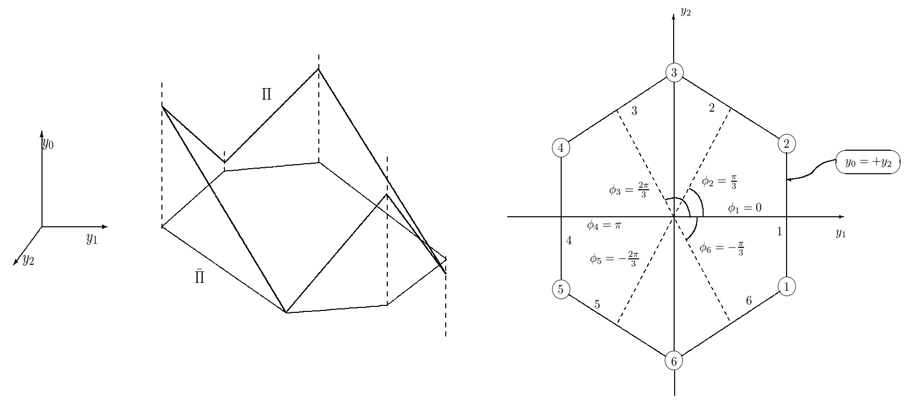

Figure 1: Convention for labeling sides and vertices of the Z_n-symmetric polygon $\bar{\Pi}$ (right). Its corresponding Π is shown in the left picture.

So far we imposed only one of the constraints (1.2), $y_3 = 0$. The second constraint can be imposed only if all $h_a = 1$, and this is what we assume below in the present paper. As already stated, this condition implies the existence of a unit circle, inscribed into $\bar{\Pi}$. If additionally n is even and $\sigma_a = (-)^a$, then also all the n parameters y_{0a} coincide and we can put $y_{0a} = 0$ by a constant shift of y_0: this choice corresponds to y_0 vanishing at all points where the sides of $\bar{\Pi}$ touch the inscribing circle.

2.3 Boundary conditions as sum rules

A possible way to describe above boundary conditions is also to write them down for each particular side of Π. On (2.4) we have (with $h_a = 1$ and $y_{0a} = 0$):

$$\begin{aligned} y_1 &= c_a - s_a t_a, \\ y_2 &= s_a + c_a t_a, \\ y_0 &= \sigma_a t_a \end{aligned} \tag{2.7}$$

i.e.

$$z = y_1 + iy_2 = (c_a + is_a)(1 + it_a) = e^{i\phi_a}(1 + i\sigma_a y_0), \tag{2.8}$$

where t_a is a parameter along the corresponding straight line. Along its segment, which is the side of Π it changes within some region $t_a \in [-t_{a1}, +t_{a2}]$. Then boundary conditions imply that

$$\sum_{i,j\geq 0} a_{ij}(c_a - s_a t_a)^i (s_a + c_a t_a)^j = \sigma_a t_a, \quad t_a \in [-t_{a1}, +t_{a2}], \quad a = 1, \ldots, n \tag{2.9}$$

A set of sum rules arise is we consider these equalities as term-by-term identities for series in powers of t_a. For example, if coordinate system is rotated to put $c_1 = 1$, $s_1 = 0$, we get an infinite set of relations

$$\sum_{j\geq 0} a_{ij} = \sigma_1 \delta_{i1} \tag{2.10}$$

– to be supplemented by $(n-1)$ more similar sets, associated with other sides of Π. The free parameters a_{k0} and a_{0k} are defined by boundary conditions.

Among other things, this consideration seems to imply that y_0 should satisfy (2.4) along entire straight line, not only within the segment. This is consistent with the known property of solutions to Plateau problem in the flat Euclidean space [33].

2.4 Approximate methods

Unfortunately no way is known at the moment to solve above relations exactly, except for in a few simple situations, listed in s.2 of [1]. In order to proceed one is naturally turned to approximate considerations.

However, there are no ready methods to address this kind of problems and one needs to practice the *trial and error* approach.

Usually approximation starts from making the best thinkable anzatz, explicitly taking into account all the already known properties of the problem (symmetries, to begin with) with remaining infinite-parameter freedom contained in adequately defined formal series. Then this anzatz is substituted into original equation, and – if the formal series was introduced in an adequate way (what is more a matter of art or lack than of a rigorous theory), – the equation turns into a *recurrent* relation for coefficients of the series. So far everything was exact, even if not fully deductive, approximation comes at the next stage: when infinite series is *truncated* at some level N. Success of the method depends on the choice of "original knowledge", of particular anzatz, including a point to expand around and particular expansion parameters, and – not the least – on the properties of the problem, i.e. the very existence of effective truncations, producing reasonable approximation at sufficiently low N.

In [1] various attempts were described to apply this procedure, and some of them seem relatively successful. The main problem appears to be a balance between reasonable introduction of formal series in *local* parameters (say, coordinates y) consistent with differential nature of NG equations, and adequate imposition of *global* boundary conditions, relatively far from expansion point. It turns out that, somewhat unusually, the balance should better be shifted towards the boundary conditions.

2.5 The goal of this paper

Success of [1] was due to construction of specific polynomials, named $\mathcal{P}_n(y_0, y_1, y_2)$, which had four important properties:

• $\mathcal{P}_\Pi \in \mathcal{R}_\Pi$ was an element of the boundary ring, thus an anzatz

$$\mathcal{P}_\Pi = 0 \tag{2.11}$$

satisfies boundary conditions exactly and it can be further generalized (perturbed) to

$$\mathcal{P}_\Pi = P_2\mathcal{B} \tag{2.12}$$

which continue to satisfy boundary conditions with any perturbation function $\mathcal{B}(y_0; y_1, y_2)$.[3]

• $\mathcal{P}_n$ was linear in y_0,

$$\mathcal{P}_\Pi = y_0 Q_\Pi(y_1, y_2) - K_\Pi(y_1, y_2) \tag{2.13}$$

what allowed to resolve (2.11) and treat it as an anzatz for a *single-valued* function

$$y_0^{(0)}(y_1, y_2) = \frac{K_\Pi(y_1, y_2)}{Q_\Pi(y_1, y_2)} \tag{2.14}$$

After that (2.12) could be solved iteratively, *a la* [34], and provides a formal series perturbation of this function.

• The polynomial Q_Π in (2.13) did not have zeroes *inside* $\bar{\Pi}$, in particular, it did not vanish at the origin,

$$Q_\Pi(y_1, y_2) = 1 + O(y_1, y_2), \tag{2.15}$$

what made the function (2.14) free of singularities, and this property was inherited by all perturbative corrections implied by (2.12).[4]

• The polynomial $K_\Pi(y_1, y_2)$ satisfied NG equations in the first approximation, i.e. application of NG operator provided only terms with higher powers of y_1, y_2 than were present in K_Π. This property was easy to formulate in [1] because Z_n-symmetric K_Π considered there were homogeneous polynomials (of degree $n/2$), but it becomes a subtler concept in generic situation. Still it is this property that allows to honestly treat $\mathcal{B}$ as a *perturbation*, needed to *correct* (2.14) in order to make it satisfying the NG equations.[5]

---

[3] Additional Z_n symmetry, assumed in [1], allowed to put $\mathcal{B}(y_0; y_1, y_2) = y_0 B(y_1, y_2)$, but this is not the case generically.

[4] A little care is needed at this point if one wishes to include the y_0-linear terms from $\mathcal{B}(y_0, y_1, y_2)$ into *denominators* of perturbation series, i.e. sum up the corresponding parts of the series exactly, what can always be done.

[5] Note that this approach is somewhat unusual, because it shifts emphasize from differential equations to boundary conditions. Ref.[1] describes in length how this shift of accents occurs, here we use this modified view from the very beginning. Still, it deserves reminding that one of the reasons for it is that the modern opinion is that Plateau problem arises in string/gauge duality in a special context: we need minimal surfaces in AdS space with boundaries lying at its boarder (infinity or the origin, depending on parametrization of AdS), so that their areas are diverging near the boundary. What we need are regularized areas, but regularization requires *exact* knowledge of behavior at the boundary, i.e. of allowed *type* of asymptotics – in order to define physical quantities, which are independent of the *coefficients* in front of these asymptotical terms. This is what makes care about the boundary conditions the first priority. If they are taken into account in exact way, then one can always deal with equations *a posteriori*, by minimizing the resulting regularized area over remaining free parameters, which could otherwise be fixed *a priori* by exactly solving the original equations. As explained in [13], this approach can be much simpler and more practical.

Thus in this paper our primary goal is to search for an analogue of the polynomials $\mathcal{P}_\Pi$ in the case of generic Π with n angles and inscribed circle in $\bar{\Pi}$. If they are constructed, then we can look at approximations to minimal surfaces provided by (2.14) and consider the actual role of corrections, which are obligatory non-vanishing, since $\mathcal{B} = 0$ is inconsistent with NG equations. Actually, the present paper is only a step in this direction. We begin by constructing the theory from the very beginning, but leave many important branches of possible development only mentioned, what finally prevents us from providing an exhaustive answer. Thus *de facto* the goal is to describe the context, what opens a lot of room for improvements and for getting better and wider results.

2.6 Plan of the paper

Our first subject in s.3 is conversion of NG equations into recurrence relations. Such conversion can be made over different "backgrounds", the c- and b-series of [1] being particular examples. In s.3 we concentrate on the "basic" example, with background zero, so that all other sets of recurrence relations can be considered as subalgebras of this main one. Our main interest here is deviation from harmonic functions due to the difference between non-linear NG operator and linear Laplace in one complex dimension – on the (y_1, y_2) plane.

The next s.4 addresses the problem of sharp angles – an important issue for applications in Alday-Maldacena program, because angles are the sources of most important quadratic divergencies of regularized actions. We explain how sharp-angle conditions can be formulated analytically. Of course, elements of the polygon boundary rings satisfy these conditions, but they are of course violated by generic solutions to NG equations, exact or approximate, before boundary conditions are imposed. Moreover, if boundary conditions are matched approximately, not exactly (like some options considered in [1]), one still has an opportunity to require that angles are sharp (not smoothened) – and it is here that these analytical formulas are especially useful.

In s.5 we address the problem of NG solutions for generic quadrilaterals. Despite it is solved in [13, 23], solution is not found in the form of explicit function $y_0(y_1, y_2)$. A way to bring it to such form is provided by technique of non-linear algebra [35, 36]. We demonstrate that at $n = 4$ this $y_0(y_1, y_2)$ is always a solution to an explicit *quadratic* equation, like it turned out to be in the particular case of rhombi [1].

The following subject in s.6 is boundary rings for polygons Π. The main puzzle here is the structure behind the polynomials K_Π in eq.(2.13). In [1] they were obtained from somewhat mysterious manipulations with P's from (2.6) and were found to have a form, which is very similar to (2.6) in Z_n-symmetric situation with even n:

$$K_{n/2} \sim \prod_{a=1}^{n/2} \Big(s_a y_1 + c_a y_2 \Big) \tag{2.16}$$

The problem is that this time the product at the r.h.s. is only over a half of segments and thus can not be immediately generalized to asymmetric cases (going from even to odd n introduces additional problem: the simplest choice of $\sigma_a = (-)^{a-1}$ can not be made). We demonstrate in s.6 how such polynomials can actually be constructed – though they probably do not play the same role as they did in [1]. The reason is that already in the first non-trivial asymmetric configuration – at $n = 4$ – exact solution is associated with the boundary ring element, which is not *linear*, but *quadratic* in y_0, see s.2.6 of [1]. This is the first signal that the proper analogue of $\mathcal{P}_\Pi$ in asymmetric case should not be linear. At the same time, s.5 demonstrates that *quadratic* can be enough, at least at $n = 4$ it is the case. It is still unclear what the situation is going to be beyond for $n > 4$, where explicit solutions of NG equations are yet unknown. A promising option is to look for the adequate anzatze among the boundary ring elements of order $n/2$ in y_0. According to the strategy, outlined in [13] and [1] we suggest to parameterize potentially relevant elements of the boundary rings by *a few* parameters, and treat them as if they were *moduli* of NG solutions, i.e. evaluate the regularized action and minimize it w.r.t. these parameters. This approach can finally turn simpler then direct solution of NG equations by methods, considered in s.3.

Appendix at the end of the paper contains some remarks about sophisticated notations used throughout the text.

3 NG equations as recurrence relations

The first recurrence relations were already found in [1]. It will be more convenient to switch to the complex coordinates $z = y_1 + \mathrm{i}y_2$, $\bar{z} = y_1 - \mathrm{i}y_2$ in the (y_1, y_2) plane and write instead of (2.1)

$$y_0 = \sum_{k,j\geq 0} \left(\alpha_{kj} z^k + \bar{\alpha}_{kj} \bar{z}^k\right) (z\bar{z})^j = \sum_{k,j\geq 0} \mathrm{Re}\left(\alpha_{kj} z^k\right) (z\bar{z})^j \tag{3.1}$$

3.1 Reduced NG action

Recurrent relations result from substitution of a formal series representation for $y_0(y_1, y_2)$ into NG equations, which for $y_3 = 0$ have the form

$$\begin{aligned} \frac{\partial}{\partial y_1}\left(\frac{\partial y_0}{\partial y_1}\frac{H_{22}}{r^2 L_{NG}}\right) + \frac{\partial}{\partial y_2}\left(\frac{\partial y_0}{\partial y_2}\frac{H_{11}}{r^2 L_{NG}}\right) - \frac{\partial}{\partial y_1}\left(\frac{\partial y_0}{\partial y_2}\frac{H_{12}}{r^2 L_{NG}}\right) - \frac{\partial}{\partial y_2}\left(\frac{\partial y_0}{\partial y_1}\frac{H_{12}}{r^2 L_{NG}}\right) = 0, \\ \frac{\partial}{\partial y_1}\left(\frac{\partial r}{\partial y_1}\frac{H_{22}}{r^2 L_{NG}}\right) + \frac{\partial}{\partial y_2}\left(\frac{\partial r}{\partial y_2}\frac{H_{11}}{r^2 L_{NG}}\right) - \frac{\partial}{\partial y_1}\left(\frac{\partial r}{\partial y_2}\frac{H_{12}}{r^2 L_{NG}}\right) - \frac{\partial}{\partial y_2}\left(\frac{\partial r}{\partial y_1}\frac{H_{12}}{r^2 L_{NG}}\right) + \frac{2L_{NG}}{r} = 0 \end{aligned} \tag{3.2}$$

where

$$H_{ij} = \frac{-\frac{\partial y_0}{\partial y_i}\frac{\partial y_0}{\partial y_j} + \frac{\partial r}{\partial y_i}\frac{\partial r}{\partial y_j} + \delta_{ij}}{r^2} \tag{3.3}$$

and

$$L_{NG} = \sqrt{\det_{ij} H_{ij}} = \sqrt{H_{11}H_{22} - H_{12}^2} \tag{3.4}$$

After substitution of (1.2) the two equations become dependent and we can consider any one of them. Even more convenient is to make the substitution (1.2) directly in NG action, then it depends on a single function $y_0(y_1, y_2)$ and looks like [1]

$$\int L_{NG} dy_1 dy_2 = \int \sqrt{\frac{(y_i\partial_i y_0 - y_0)^2 - (\partial_i y_0)^2 + 1}{\left(1 + y_0^2 - y_1^2 - y_2^2\right)^3}}\, dy_1 dy_2 \tag{3.5}$$

3.2 Linear approximation to NG equation and its generic solution

Equations (3.3) are highly non-linear in y_0 and it is convenient to begin with their y_0-linear approximation. Expanding (3.5) in powers of y_0, we obtain

$$\int \frac{dy_1 dy_2}{(1 - y_1^2 - y_2^2)^{3/2}} - \frac{1}{2}\int \left(\frac{(\partial_i y_0)^2 - (y_i\partial_i y_0 - y_0)^2}{(1 - y_1^2 - y_2^2)^{3/2}} + \frac{3y_0^2}{(1 - y_1^2 - y_2^2)^{5/2}}\right) dy_1 dy_2 + O(y_0^4) \tag{3.6}$$

The first (divergent) term is non-essential for equations of motion. The y_0-quadratic term gives rise to y_0-linear approximation to equations (3.3) in the simple form:

$$\Delta y_0 = 0 \tag{3.7}$$

where

$$\Delta = \Delta_0 - \mathcal{D}^2 + \mathcal{D} \tag{3.8}$$

is expressed through the ordinary Laplace

$$\Delta_0 = \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}} = 4\partial\bar{\partial} \tag{3.9}$$

and dilatation operators

$$\mathcal{D} = y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} = z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}} = z\partial + \bar{z}\bar{\partial} \tag{3.10}$$

If there were no dilatation operators in (3.8), like it happens in the flat R^3 space (i.e. if we linearize not only w.r.t. y_0 but also w.r.t. y_1 and y_2), then the solution of the ordinary Laplace equation $\Delta_0 y_0^{flat} = 0$ would be just a combination of holomorphic and antiholomorphic functions,

$$y_0^{flat} = \sum_{k\geq 0} \mathrm{Re}\left(\alpha_{k0} z^k\right) \tag{3.11}$$

However, in AdS_3 case the situation is different: $\alpha_{kj} \neq 0$ for all $j \neq 0$ in (3.1). Substitution of (3.1) into (3.7) gives rise to linearized version of recurrence relations,

$$\alpha_{k,j+1}^{lin} = \frac{(k+2j)(k+2j-1)}{4(j+1)(k+j+1)}\,\alpha_{kj}^{lin}, \tag{3.12}$$

which can be easily resolved to give:

$$\alpha_{kj} = \frac{k!(k+2j-2)!}{4^j j!(k-2)!(k+j)!}\,\alpha_{k0} + O(\alpha^3) = \frac{k(k-1)}{4^j j!}\frac{(k+2j-2)!}{(k+j)!}\,\alpha_{k0} + O(\alpha^3) \tag{3.13}$$

(One easily recognizes here eq.(4.2) of [1] with $k = n/2$.) Therefore in linear approximation

$$y_0^{lin} = \sum_{k\geq 0} \mathrm{Re}\left(\alpha_{k0} z^k\right)\cdot\ {}_2F_1\left(\frac{k}{2}, \frac{k-1}{2}; k+1;\, z\bar z\right) \tag{3.14}$$

is a combination of hypergeometric functions

$${}_2F_1(a,b;c;x) = \sum_{j\geq 0} \frac{\Gamma(a+j)\Gamma(b+j)}{j!\Gamma(c+j)}\,x^j \tag{3.15}$$

3.3 Back to non-linear NG equations

The full non-linear NG equation, implied by (3.5), can be written in a form, which looks like a deformation of (3.7):

$$\left\{\Big(1 + y_0^2 + (y^2-1)(\partial_i y_0)^2 - 2y_0(\mathcal{D}y_0)\Big)\Delta_0 - \mathcal{D}^2 + \mathcal{D} + \Big((1-y^2)\partial_i y_0 + 2y_i y_0\Big)\partial_j y_0\,\partial_{ij}^2\right\} y_0 = 0 \tag{3.16}$$

where $y^2 = y_1^2 + y_2^2$ and $i, j = 1, 2$, or, in complex notation,

$$\begin{aligned}\Big\{\Big(1 + y_0^2 + 2(z\bar z - 1)\partial y_0 \bar\partial y_0 - y_0\mathcal{D}y_0\Big)\partial\bar\partial - \frac{1}{4}(\mathcal{D}^2 - \mathcal{D}) + \\ + \Big((1-z\bar z)(\bar\partial y_0)^2 + z y_0 \bar\partial y_0\Big)\partial^2 + \Big((1-z\bar z)(\partial y_0)^2 + \bar z y_0 \partial y_0\Big)\bar\partial^2\Big\}\, y_0 = 0\end{aligned} \tag{3.17}$$

If all terms with y_0 in curved brackets are neglected, we return back to (3.7). Note that the equation is at most cubic in y_0, what implies that it can be obtained also from some ϕ^4-type action, somewhat less non-linear than NG one.

If equation (3.17) is solved iteratively, it gives rise to more sophisticated recurrence relations. In order to obtain them we rewrite (3.17) as $\Delta y_0 = 4h$, where h is formed by all y_0-cubic terms in (3.17). Then instead of (3.12) we get

$$\alpha_{k,j+1}^{(h)} = \frac{(k+2j)(k+2j-1)}{4(j+1)(k+j+1)}\,\alpha_{kj}^{(h)} + \frac{1}{(j+1)(k+j+1)}\,h_{kj}, \tag{3.18}$$

At the next stage h_{kj} are substituted by cubic combinations of $\alpha_{k'j'}$ with lower values of k' and $j\,'$ and this provides cubic recurrence relations for α_{kj}, which we do not write down explicitly in this paper.

4 Angles in the case of approximately imposed boundary conditions

4.1 Approximation can damage IR properties of the regularized action

Before we proceed in s.6 to construction of the boundary ring $\mathcal{R}_\Pi$ for a given polygon Π, consider a reversed problem: how can polygon Π be defined by a pair of algebraically independent elements from $\mathcal{R}_\Pi$, say

$$\Pi = \left\{ \begin{array}{l} P_2 = 0, \\ P_\Pi = 0 \end{array} \right. \tag{4.1}$$

There are only two equations because we assume that there are just three y-variables, i.e. $y_3 = 0$. For one of these equations one can always take $P_2 = 0$ because we assume existence of inscribed circle and thus of distinguished element $P_2 \in \mathcal{R}_\Pi$. In some of our approximate considerations we actually substitute the second equation $P_\Pi = 0$ by some truncated series for y_0, $y_0 - F(y_1, y_2) = 0$, which does not belong to $\mathcal{R}_\Pi$. Thus instead of Π we obtain some approximation:

$$\tilde{\Pi} = \left\{ \begin{array}{c} P_2 = y_0^2 + 1 - y_1^2 - y_2^2 = 0, \\ y_0 = F(y_1, y_2) \end{array} \right. \tag{4.2}$$

and instead of $\bar{\Pi}$ – a curve on the $y_1, y_2)$ plane

$$\bar{\tilde{\Pi}} = \Big\{ G_\Pi(y_1, y_2) = 0 \Big\} \tag{4.3}$$

In the case of (4.2) this

$$G_\Pi(y_1, y_2) = F^2(y_1, y_2) + 1 - y_1^2 - y_2^2, \tag{4.4}$$

but even if the second equation in (4.2) is not explicitly resolved w.r.t. y_0, there will be a polynomial $G_\Pi(y_1, y_2)$, defining $\bar{\tilde{\Pi}}$.

Of course, in approximate treatment $\bar{\tilde{\Pi}}$ is no longer a polygon, actually, for two reasons: it is not made from straight segments and it does not contain *angles*, generically $G = 0$ is a smooth curve. The latter deviation from polygonality can be most disturbing for applications, like string/gauge duality, which involve consideration of areas of our minimal surfaces. Since in our approach

$$r^2 = P_2 \stackrel{(4.4)}{=} G_2(y_1, y_2), \tag{4.5}$$

the area in question is

$$\mathcal{A} = \int L_{NG} d^2y = \int_{G>0} \frac{H d^2y}{G} \tag{4.6}$$

with some non-singular function $H(y_1, y_2)$ in denominator. This integral diverges at the boundary of integration domain, where $G = 0$, but this is generically a logarithmic divergence: if integral is regularized in any of the two obvious ways,

$$\mathcal{A}[\varepsilon] = \int_{G>\varepsilon} \frac{H d^2y}{G} \tag{4.7}$$

or

$$\mathcal{A}(\epsilon) = \int_{G>0} \frac{H d^2y}{G^{1-\epsilon}}, \tag{4.8}$$

to be called ε- and ϵ-regularizations in what follows, we generically get

$$\mathcal{A}[\varepsilon] \sim \log \varepsilon \oint \sqrt{h^{[\varepsilon]}} dl + A_{finite}^{[varepsion]} \tag{4.9}$$

or

$$\mathcal{A}(\epsilon) \sim \frac{1}{\epsilon} \oint \sqrt{h^{(\epsilon)}} dl + A_{finite}^{(varepsion)} \tag{4.10}$$

However, if the resulting metrics h are themselves singular, divergence can become quadratic, and this is what actually happens if the curve $\tilde{\tilde{\Pi}}: \ G = 0$ is singular: has *angles*. Then additional terms,

$$\sum_{angles} (\log\varepsilon)^2 \cdot \kappa(angle) \quad \text{and} \quad \sum_{angles} \frac{1}{\epsilon^2}\cdot\kappa(angle) \tag{4.11}$$

appear at the r.h.s. of (4.9) and (4.10) respectively. Since $\kappa(angle) \sim \sin(angle)$, smoothening of the curve has a drastic effect on divergencies of regularized area, which are interpreted as IR singularities in string/gauge duality studies. This smoothening can be of course actually considered as an alternative (or, rather, supplementary) regularization, but using it can further obscure the problem, which is already sufficiently complicated. Instead one can require that the angles – sources of dominant (quadratic) IR divergencies – are preserved by our approximate schemes. This imposes a new kind of restrictions on the free parameters of formal series solutions and provide an alternative way to fix some of them (which gives values slightly different from other approaches).

4.2 Angles and discriminants

Singularities in algebraic geometry are analytically described in terms of discriminants and resultants, see [35, 36] for a modernized presentation of these methods, of which only a standard elementary part will be used in this paper.

The curve $G(y_1, y_2) = 0$ possesses angles whenever repeated discriminant vanishes,

$$\text{discrim}_{y_2}\Big(\text{discrim}_{y_1}\Big(G(y_1,y_2)\Big)\Big) = 0 \tag{4.12}$$

Indeed, as a function of y_1 the polynomial $G(y_1, y_2)$ can be decomposed into a product

$$G(y_1,y_2) = \prod_\nu \Big(y_1 - \lambda_\nu(y_2)\Big) \tag{4.13}$$

Each eigenvalue $\lambda_\nu(y_2)$ describes a branch of our curve. Branches intersect whenever the two eigenvalues coincide, i.e. when discriminant [37]

$$D(G; y_2) = \text{discrim}_{y_1}(G) \sim \prod_{\mu<\nu}\Big(\lambda_\mu(y_2) - \lambda_\nu(y_2)\Big)^2 \tag{4.14}$$

vanishes. For given function G this condition defines some points on the (y_1, y_2) plane, a variety of complex codimension one. However, there are two different situations: two branches can *merge* and they can indeed *intersect*. Merging is in the degree of discriminant's zero at the intersection. If two branches are indeed intersecting at some non-vanishing angle at a point $y_2 = y_{20}$, we expect that

$$\lambda_\mu(y_2) - \lambda_\nu(y_2) = (\lambda'_\mu - \lambda'_\nu)(y_2 - y_{20}) \tag{4.15}$$

where the difference of λ-derivatives at point y_{20} is the tangent of the intersection angle. However, this implies that discriminant in (4.14) behaves as $(y_2 - y_{20})^2$, i.e. has a *double* zero. This is not usual, normally discriminant zeroes are of the first order, then $\delta\lambda \sim \sqrt{y_2 - y_{20}}$ and the branches *merge* smoothly, tangents to the curves $y_1 = \lambda_\mu(y_1)$ and $y_2 = \lambda_\nu(y_2)$ coincide (as it happens, for example, when the two real roots of quadratic polynomial merge and then decouple into two complex conjugate ones: the difference between the two roots has a square root singularity what means that the tangents get both vertical and thus coincide!). Thus the condition that two branches intersect at non-vanishing angle, i.e. that $\tilde{\tilde{\Pi}}$ has angles, is that discriminant $D(G; y_2)$ possesses double zeroes, i.e. that *its own* discriminant vanishes:

$$\text{discrim}_{y_2}\Big(D(G;y_2)\Big) = 0 \tag{4.16}$$

This is exactly the equation (4.12) – and it is a restriction on the shape of the function $G(y_1, y_2)$.

4.3 A way to proceed in ε-regularization

Making use of decomposition (4.13), we can write

$$\frac{1}{G} = \frac{1}{\prod_i(y_1 - \lambda_i(y_2))} = \sum_i \frac{1}{y_1 - \lambda_i}\prod_{j\neq i}\frac{1}{\lambda_j - \lambda_i} \tag{4.17}$$

Divergent part of integral over y_1 is thus

$$\int \frac{dy_1}{G(y_1,y_2)} \sim \log\varepsilon \sum_i \prod_{j\neq i} \frac{1}{\lambda_j - \lambda_i} \tag{4.18}$$

and the remaining integral over y_2 diverges whenever some $\lambda_j(y_2) = \lambda_k(y_2)$, i.e. at y_2 which are roots of the discriminant $\mathrm{Disrim}_{y_1}(G)$. It is also clear that these are the *angles* of our boundary, $G(y_1, y_2) = 0$ which consists of lines $y_1 = \lambda_i(y_2)$, at intersection points they form angles, and these angles produce quadratic divergencies. Linear divergencies come from the sides (lines themselves) and we are interested in separating the finite piece.

The basic example is $G = (1-y_1^2)(1-y_2^2)$, then it is easy to observe the $(\log\varepsilon)^2$.

Similarly one can analyze ϵ-regularization.

It is unclear how to extract the finite part. Probably this could be done numerically, but for this the divergent parts should first be subtracted "by hands".

4.4 Z_n-symmetric examples

We illustrate above consideration with the help of a few examples. For the sake of simplicity we pick up the Z_n-symmetric configurations, analyzed in [1].

Plots for $y_1(y_2)$ are obtained by solving

$$P_2 = y_0^2 + 1 - y_1^2 - y_2^2 = 0 \tag{4.19}$$

with

$$y_0 = c_n K_{n/2} = 2^{1-n/2} c_n \mathrm{Im}(y_1 + \imath y_2)^{n/2} \tag{4.20}$$

The following is a small piece of calculations behind s.4.3.4 of [1].

4.4.1 $n = 4$

In this case the equation

$$G(y_1, y_2) = (cy_1y_2)^2 + 1 - y_1^2 - y_2^2 = 0 \tag{4.21}$$

is easily resolved:

$$y_1 = \pm\sqrt{\frac{y_2^2 - 1}{c^2y_2^2 - 1}} \tag{4.22}$$

and plots of this function at different values of c are shown in Fig.2. Distinguished point $c = 1$ is clearly see. In terms of discriminants we have:

$$D(G; y_2) = \mathrm{discrim}_{y_1}(G) = 4(c^2y_2^2 - 1)(y_2^2 - 1) \tag{4.23}$$

(the two branches in (4.22) merge when discriminant vanishes, either at zero or at infinity, when $y_2 = \pm c^{-1}$ and $y_2 = \pm 1$ respectively), and

$$\mathrm{discrim}_{y_2} D(G; y_2) = 65536c^2(c^2 - 1)^4 \tag{4.24}$$

(double discriminant vanishes when branches intersect: at $c = \pm 1$ they do so at four points, thus zero is of the fourth power – the vertices of our square,– while at $c = 0$ an intersection at two points takes place at infinity).

4.4.2 $n = 6$

This time the plots for $y_1(y_2)$ obtained by solving

$$P_2 = y_0^2 + 1 - y_1^2 - y_2^2 = 0 \tag{4.25}$$

with

$$y_0 = c_3K_3 = cy_2(3y_1^2 - y_2^2) \tag{4.26}$$

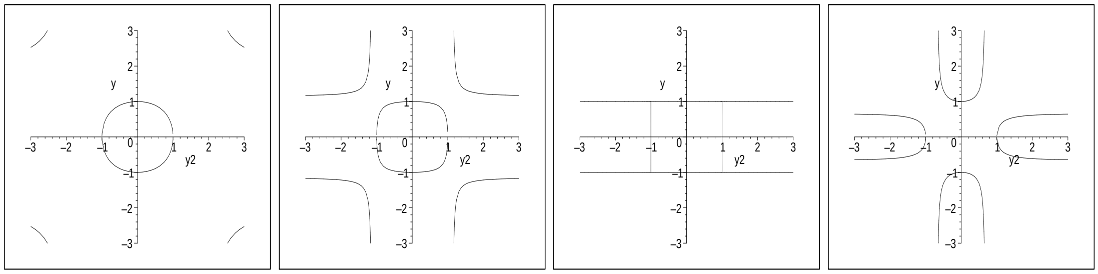

Figure 2: The plot of the function $y_1(y_2)$ in (4.22) at different values of c= 0.5, 0.87, 1 and 1.5. It is clearly seen that the unit square is formed at exactly $c = 1$, as predicted by (4.24.

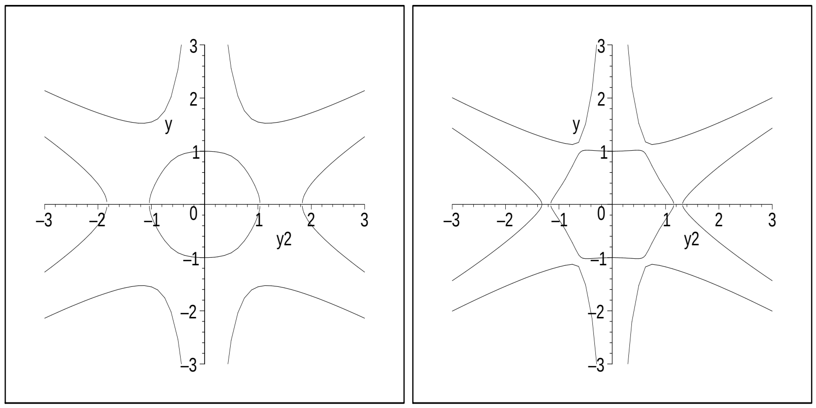

Figure 3: The plot of $y_1(y_2)$ at $c = 1/4$ (left) and at $c = \frac{2\sqrt{3}}{9} - \frac{1}{100}$ (right). In the left picture the central domain is far from being a polygon: at this value of c it looks almost like a circle (and will get even closer to this shape for smaller $|c|$). The right picture shows what happens in a close vicinity of the critical value of $c = \frac{2\sqrt{3}}{9}$. The central domain still does not possess angles, see also Fig.6, but is already close to that. Note that parameter c here is different from $c_{00}^{(6)}$ in [1]: $c = \frac{1}{4}c_{00}^{(6)}$.

so that $c = \frac{1}{4}c_3$.

Discriminant

$$\mathrm{discrim}_{y_1}(G) = 144c^2y_2^2(c^2y_2^6 - y_2^2 + 1)(48c^2y_2^4 - 36c^2y_2^2 + 1)^2 \tag{4.27}$$

Since powers appear at the r.h.s., repeated discriminant w.r.t. y_2 is vanishing and we need to look at the individual factors at the r.h.s.:

$$\begin{gathered}\mathrm{discrim}_{y_2}(c^2y_2^6 - y_2^2 + 1) = -64c^6(27c^2 - 4)^2, \\ \mathrm{discrim}_{y_2}(48c^2y_2^4 - 36c^2y_2^2 + 1) = 1769472c^6(27c^2 - 4)^2, \\ \mathrm{resultant}_{y_2}(c^2y_2^6 - y_2^2 + 1, +48c^2y_2^4 - 36c^2y_2^2 + 1) = c^8(216c^2 + 49)^4\end{gathered} \tag{4.28}$$

The interesting critical values of c are zeroes of $27c^2 - 4$, i.e. $c = \pm\frac{2\sqrt{3}}{9} = \pm 0.38490\ldots$. Figs.3-6 show exact meaning of these calculations and preceding argumentation.

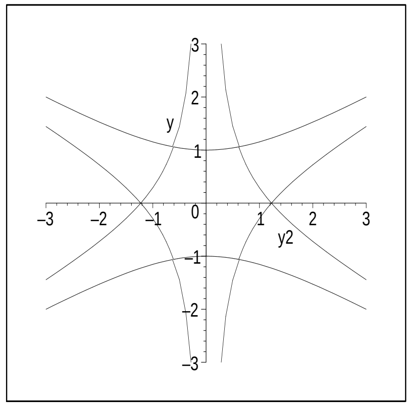

Figure 4: The plot of $y_2(y_1)$ at the critical value of $c = \frac{2\sqrt{3}}{9}$. Angles are well seen at the intersections of different branches, the central domain looks similar to a hexagonal polygon. Despite angles exist, the sides are not exactly straight: (4.26) satisfies boundary conditions (and also NG equations) only approximately, this value of c is distinguished by existence of angles.

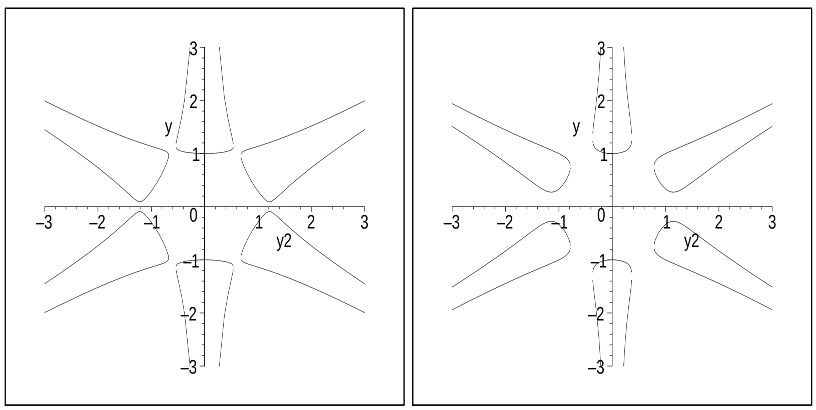

Figure 5: The plot of $y_1(y_2)$ at $c = \frac{2\sqrt{3}}{9} + \frac{1}{100}$ (left), in close vicinity of the critical value of $\frac{2\sqrt{3}}{9}$, and at $c = \frac{1}{2}$ (right), a little further away. Different branches are now intersecting at complex values y-variables, and the central domain is no longer closed.

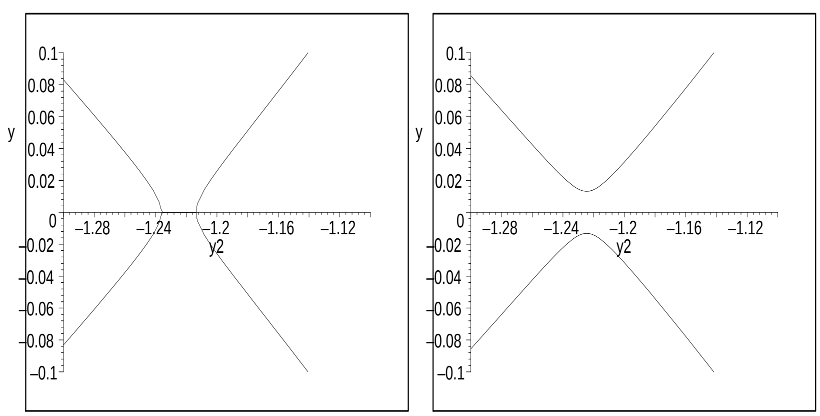

Figure 6: Enlarged pictures, showing the vicinity of the branches merging point at $c = \frac{2\sqrt{3}}{9} - \frac{2}{1000}$ (left picture) and $c = \frac{2\sqrt{3}}{9} + \frac{2}{1000}$ (right picture) – i.e. at the very close vicinity of the critical point $c = \frac{2\sqrt{3}}{9}$. Clearly, no angles are present at "microscopic" level. They appear exactly at the critical point, where the two branches intersect.

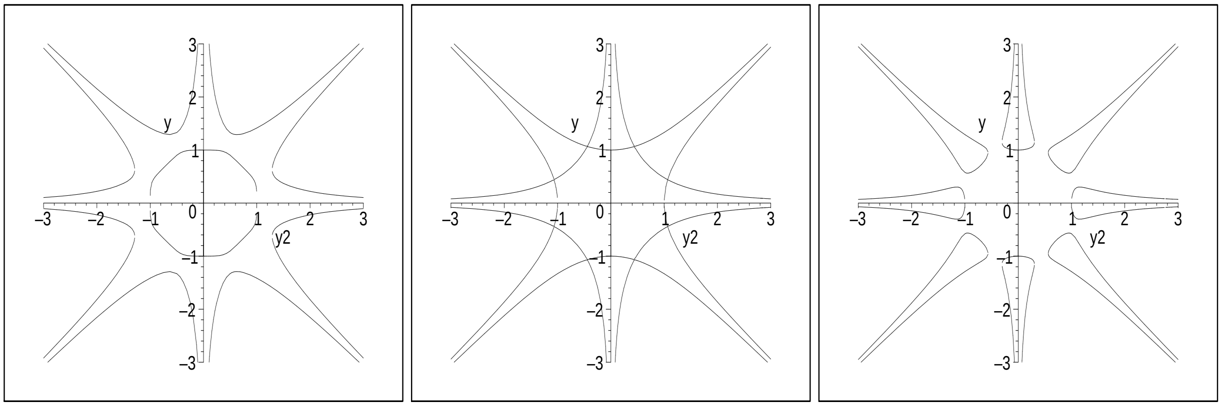

Figure 7: The analogues of Figs.3-5 for $n = 8$ with $y_0 = cK_4 = cy_1y_2(y_1^2 - y_2^2)$ at $c = 1$, $c = \frac{3\sqrt{3}}{4}$ (the critical value) and $c = \frac{3}{2}$. At critical value the branches intersect at non-trivial angles, but the sides of emerging octagon are not straight: boundary conditions (and NG equations) are matched only approximately. The sides look "more straight" in the left picture – for c below the critical point, where angles are less pronounced: this illustrates the thesis that different criteria lead to slightly different values of the matching parameter c. This choice of parameter is different from $c_{00}^{(8)}$ in [1]: $c = \frac{1}{2}c_{00}^{(8)}$.

4.4.3 $n = 8$

This time

$$y_0 = c_4K_4 = cy_1y_2(y_1^2 - y_2^2), \tag{4.29}$$

so that $c = \frac{1}{2}c_4$, the plots for $y_2(y_1)$ are shown in Fig.7 and discriminants are:

$$\begin{gathered}\mathrm{discrim}_{y_1}(G) = 64c^6y_2^6(y_2^2 - 1)g^2(y_2, c),\\ g(y_2, c) = 4 - 27c^2y_2^2 + 90c^2y_2^4 - 71c^2y_2^6 - 4c^4y_2^{10} + 8c^4y_2^{12},\\ \mathrm{discrim}_{y_2}g = 137438953472c^{44}(16c^2 - 27)^4(243c^2 + 4913)^6\end{gathered} \tag{4.30}$$

so that the relevant zero is $c = \frac{3\sqrt{3}}{4}$.

4.5 Exact solutions to (4.12)

The angle-sharpening problem can actually be reversed: one can consider (4.12) as an equation for $G(y_1, y_2)$. In [1] we already showed exact solutions to this problem:

$$G_\Pi = K_\Pi^2 + (1 - y^2)Q_\Pi^2 = P_\Pi = \prod_{\substack{\text{segments}\\ \text{of } \Pi}} P_|, \tag{4.31}$$

are totally decomposed into a product of linear functions, associated with individual segments, see (2.6). The corresponding analogues of Figs.3-7 are just 6 or 8 straight lines which form the regular hexagon and octagon at the intersection, see Fig.8. In formulas for (4.31) this looks like:

$$\begin{gathered} n = 4: \qquad (y_1y_2)^2 + (1 - y^2) = (1 - y_1)(1 + y_1)(1 - y_2)(1 + y_2),\\ n = 6: \qquad \left(\frac{y_2(3y_1^2 - y_2^2)}{4}\right)^2 + (1 - y^2)\left(1 - \frac{y^2}{4}\right)^2 =\\ = \quad (1 - y_1)(1 - cy_1 - sy_2)(1 + cy_1 - sy_2)(1 + y_1)(1 + cy_1 + sy_2)(1 - cy_1 + sy_2),\\ \text{with} \quad c = \frac{1}{2}, \quad s = \frac{\sqrt{3}}{2},\\ \ldots \end{gathered} \tag{4.32}$$

These examples are provided by the knowledge of boundary rings, their perturbation like (2.12) should give rise to more solutions and (4.12) can serve as one more property of $\mathcal{P}_\Pi$, to be added to the list in s.2.5.

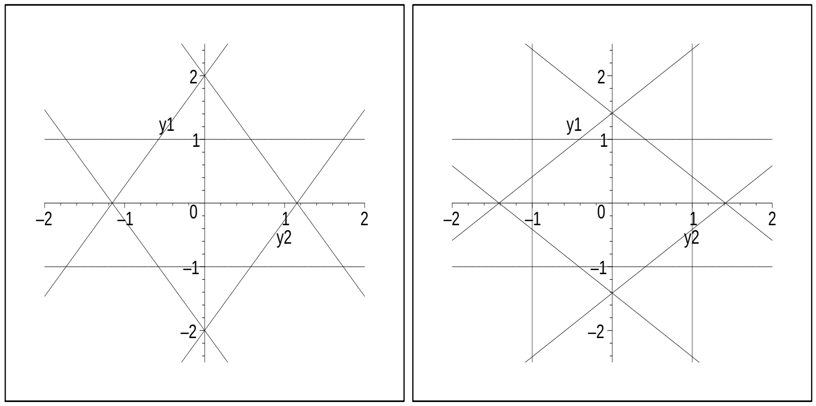

Figure 8: The analogues of Figs.3-5 for $n = 6$ with $y_0 = \frac{K_3}{(1-\frac{1}{4}y^2)}$ and of Fig.7 for $n = 8$ with $y_0 = \frac{K_4}{(1-\frac{1}{2}y^2)}$ which satisfy the boundary condition exactly. Ideal hexagon and octagon with sharp angles and straight sides are clearly seen in the pictures.

5 NG solution for generic skew quadrilateral

Solutions to the σ-model and NG equations with such boundary conditions were considered in [13] and [23] respectively. Though the single-parametric rhombus family, originally introduced in [32, 4], is sufficient for direct application to string-gauge duality studies, generic solutions are definitely interesting from the point of view of Plateau problem. The difficulty is that in [23] NG solution is not represented in the resolved form, as $y_0(y_1, y_2)$, it is left in a parametric representation, inherited from the σ-model solution of [13]. The situation is similar to the rhombic solution, which is transformed from the parametric representation of [32, 4, 13] to resolved expression only in s.2.6 of [1].

5.1 Solutions from [13, 23]

For $n = 4$ coordinate system can always be rotated so, that the boundary conditions and thus a solution (the one which does not correspond to spontaneously broken Z_2-symmetry $y_3 \to -y_3$) have $y_3 = 0$. The skew quadrilateral Π is formed by four null-vectors only provided $\bar{\Pi}$ possesses an inscribed circle, thus the conditions (1.2) can always be imposed. It is only important to remember that in this form it requires the special choice of coordinate system: $y_1 = y_2 = 0$ at the center of the circle, and $y_0 = 0$ at its tangent points with the sides of the quadrilateral (if y_0 vanishes at any of these points, it automatically does so at the other three). Thus NG solution is described by a single function $y_0(y_1, y_2)$.

In [13, 23] it is instead described in a very different way: r and $\mathbf{y} = (y_0; y_1, y_2)$ are expressed through the variables $z = 1/r$ and $\mathbf{v} = z\mathbf{y}$, which are actually the embedding (most natural) coordinates for AdS σ-model. In these variables generic solution looks simple:

$$\begin{aligned} z &= z_1(e^{\vec{k}_1\vec{u}} + e^{-\vec{k}_1\vec{u}}) + z_2(e^{\vec{k}_2\vec{u}} + e^{-\vec{k}_2\vec{u}}), \\ \mathbf{v} &= \mathbf{v}_1 e^{\vec{k}_1\vec{u}} + \mathbf{v}_3 e^{-\vec{k}_1\vec{u}} + \mathbf{v}_2 e^{\vec{k}_2\vec{u}} + \mathbf{v}_4 e^{-\vec{k}_2\vec{u}} \end{aligned} \tag{5.1}$$

Remaining parameters are constrained by NG equations and boundary conditions. The latter imply that

$$\frac{\mathbf{v}_{a+1}}{z_{a+1}} - \frac{\mathbf{v}_a}{z_a} = \mathbf{p}_a, \qquad a = 1, 2, 3, 4 \tag{5.2}$$

where $\mathbf{p}_a$ are the four null-vectors, forming the sides of our polygon Π (i.e. external momenta of the four gluons). The former imply that

$$\begin{gathered} z_1 = z_3 = \frac{1}{\sqrt{2s}} = \frac{1}{2\sqrt{\mathbf{p}_1\mathbf{p}_2}}, \quad z_2 = z_4 = \frac{1}{\sqrt{2t}} = \frac{1}{2\sqrt{\mathbf{p}_2\mathbf{p}_3}} = \frac{1}{2\sqrt{\mathbf{p}_1\mathbf{p}_4}}, \\ s = (\mathbf{p}_1 + \mathbf{p}_2)^2 = 2\mathbf{p}_1\mathbf{p}_2, \quad t = (\mathbf{p}_2 + \mathbf{p}_3)^2 = 2\mathbf{p}_2\mathbf{p}_3 \end{gathered} \tag{5.3}$$

Our usual variables are:

$$r = \frac{1}{z}, \quad \mathbf{y} = \frac{\mathbf{v}}{z} \tag{5.4}$$

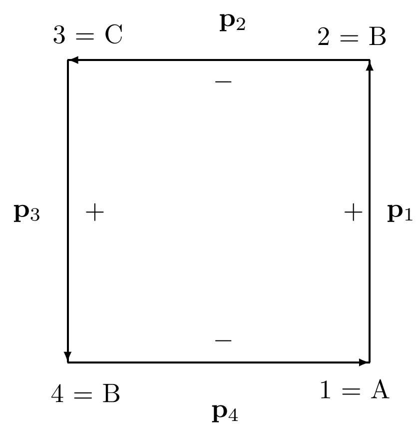

Figure 9: Convention for labeling sides and vertices of the quadrilateral, a square is used as an example. Pluses and minuses stand for y_0 increasing (+) or decreasing (-) along the vector.

5.2 From $\mathbf{y}(\vec{u})$ to $y_0(y_1, y_2)$

Our goal is to express y_0 through y_1 and y_2, i.e. to eliminate two variables $\vec{u}$ from the three-component vector equation (5.1) for $\mathbf{y} = z^{-1}\mathbf{v}$. Our strategy is to reformulate the problem in terms of polynomials and then solve it with the standard methods of *non-linear algebra* [36]. In result we obtain y_0 as a solution to *quadratic* equation, which will be afterwards compared with the results from boundary ring considerations.

Our equations become polynomial in terms of $U \equiv e^{\vec{k}_1\vec{u}}$ and $W \equiv e^{\vec{k}_2\vec{u}}$:

$$z_1(\mathbf{y}-\mathbf{y}_A)U + z_2(\mathbf{y}-\mathbf{y}_B)W + z_1(\mathbf{y}-\mathbf{y}_C)U^{-1} + z_2(\mathbf{y}-\mathbf{y}_D)W^{-1} = \mathbf{0} \tag{5.5}$$

where the four vertices are now denoted by A, B, C, D, see Fig.9, and $\mathbf{y}_a = \frac{\mathbf{v}_a}{z_a}$, with $a = A, B, C, D$, $z_A = z_C = z_1$, $z_B = z_D = z_2$, are the values of $\mathbf{y}$ at these vertices. Of course, resolvability of the system (5.5) in four variables U, U^{-1}, W, W^{-1} requires that the 4×4 determinant vanishes – and this is guaranteed by the possibility to choose all 3-components of $\mathbf{y}$ and $\mathbf{y}_a$ vanishing, so that vectors in (5.5) have only three components, $0, 1, 2$. However, since of the four variables U, U^{-1}, W, W^{-1} only two are algebraically independent the vanishing of 4×4 determinant is not the only resolvability condition. The more restrictive discriminantal constraint can be derived as follows.

Take any pair of the three equations in (5.5) and eliminate W^{-1} or W:

$$\begin{aligned} z_1\Big(\mathbf{K}_{AD}U + \mathbf{K}_{CD}U^{-1}\Big) + z_2\mathbf{K}_{BD}W &= \mathbf{0}, \\ z_1\Big(\mathbf{K}_{AB}U - \mathbf{K}_{BC}U^{-1}\Big) - z_2\mathbf{K}_{BD}W^{-1} &= \mathbf{0} \end{aligned} \tag{5.6}$$

Here $K^{\lambda}_{ab} = \epsilon^{\lambda\mu\nu}K^{\mu\nu}_{ab}$ with

$$K^{\mu\nu}_{ab} = (y^\mu - y^\mu_a)(y^\nu - y^\nu_b) - (y^\mu - y^\mu_b)(y^\nu - y^\nu_a) = y^\mu(y^\nu_a - y^\nu_b) + y^\nu(y^\mu_b - y^\mu_a) + (y^\mu_a y^\nu_b - y^\mu_b y^\nu_a) \tag{5.7}$$

and $\lambda, \mu, \nu = 0, 1, 2$ is *linear* in y-variables and antisymmetric in ab.

Picking any component of the first and any component of the second equation in (5.6) we can use $WW^{-1} = 1$ to obtain *nine* equations:

$$z_2^2\mathbf{K}_{BD}\otimes\mathbf{K}_{BD} = z_1^2\Big(\mathbf{K}_{AD}U + \mathbf{K}_{CD}U^{-1}\Big)\otimes\Big(\mathbf{K}_{BC}U - \mathbf{K}_{AB}U^{-1}\Big) \tag{5.8}$$

or

$$z_1^2\mathbf{K}_{AD}\otimes\mathbf{K}_{AB}U^4 + \Big(z_1^2\mathbf{K}_{CD}\otimes\mathbf{K}_{AB} - z_1^2\mathbf{K}_{AD}\otimes\mathbf{K}_{BC} + z_2^2\mathbf{K}_{BD}\otimes\mathbf{K}_{BD}\Big)U^2 - z_1^2\mathbf{K}_{CD}\otimes\mathbf{K}_{BC} = \mathbf{0}\times\mathbf{0} \tag{5.9}$$

Consistency of any pair of these equations is a non-trivial condition on $\mathbf{K}$ (all 36 pairs are giving rise to equivalent $y_0(y_1, y_2)$!). According to [36],

$$\sum_{\beta,\gamma=\pm}^{2} T_{\alpha\beta\gamma}x_\beta x_\gamma = 0 \tag{5.10}$$

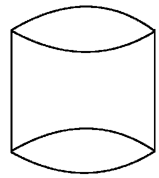

Figure 10: Feynman diagram for the Cayley hyperdeterminant (5.11). Tensor $T_{\alpha\beta\gamma}$ stands at the valence-three vertices, while propagators are ϵ-symbols. See [36] for more explanations.

is resolvable system of two equations (with $\alpha = 1, 2$) for two variables x_+, x_- iff its resultant $R_{2|2}$ – which in this case coincides with the Cayley discriminant or "hyperdeterminant" [38], see Fig.10,– vanishes:

$$D_{2|3}(T) = \varepsilon^{\alpha\alpha''}\varepsilon^{\alpha'\alpha'''}\epsilon^{\beta\beta'}\epsilon^{\gamma\gamma'}\epsilon^{\beta''\beta'''}\epsilon^{\gamma''\gamma'''}T_{\alpha\beta\gamma}T_{\alpha'\beta'\gamma'}T_{\alpha''\beta''\gamma''}T_{\alpha'''\beta'''\gamma'''} =$$
$$= (T_{1++}T_{2--} - T_{1--}T_{2++})^2 + 4(T_{1+-}T_{2++} - T_{1++}T_{+-})(T_{1+-}T_{2--} - T_{1--}T_{2+-}) = 0 \qquad (5.11)$$

Of course, this is nothing but the condition that two quadratic equations have a common root and can be derived by elementary means, say, from explicit knowledge of the formula for the roots. In our case $x_+ = U^2$, $x_- = 1$, and tensor $T_{\alpha\beta\gamma}$ is made out of $\mathbf{K}\otimes\mathbf{K}$. Discriminant $D_{2|3}$ is bilinear in both components of $T_{1..}$ and $T_{2..}$, while K^λ is linear in the complementary y-variables (i.e. in y^μ with $\mu \neq \lambda$. Thus discriminantal condition can be made quadratic in y_0 if we choose as a pair of equations from (5.9) either K^0K^1 and K^0K^0 or K^0K^2 and K^0K^0. Indeed, K^0 is independent of y_0, while K^1 and K^2 are linear in y_0, thus the corresponding discriminants will be quadratic. Instead, both expressions are *a priori* asymmetric in y_1 and y_2, one can also consider a linear combination $K^0(\mu K^1 + \nu K^2)$ to put this asymmetry under control.

Example: In the case of the **square** we have, see Fig.9:

$$\mathbf{p}_1 = (2; 0, 2), \quad \mathbf{p}_2 = (-2; -2, 0), \quad \mathbf{p}_3 = (2; 0, -2), \quad \mathbf{p}_4 = (-2; 2, 0), \qquad z_1 = z_2 = \frac{1}{4} \qquad (5.12)$$

and

$$\mathbf{y}_A = (-1; 1, -1), \quad \mathbf{y}_B = (1; 1, 1), \quad \mathbf{y}_C = (-1; -1, 1), \quad \mathbf{y}_D = (1; -1, -1), \qquad (5.13)$$

so that

$$y_0 = \frac{-U + W - U^{-1} + W^{-1}}{U + W + U^{-1} + W^{-1}},$$
$$y_1 = \frac{U + W - U^{-1} - W^{-1}}{U + W + U^{-1} + W^{-1}},$$
$$y_2 = \frac{-U + W + U^{-1} - W^{-1}}{U + W + U^{-1} + W^{-1}} \qquad (5.14)$$

These equations are simple enough to be solved directly:

$$U = \sqrt{\frac{(1+y_1)(1-y_2)}{(1-y_1)(1+y_2)}}, \qquad W = \sqrt{\frac{(1+y_1)(1+y_2)}{(1-y_1)(1-y_2)}} \qquad (5.15)$$

and in this case $y_0(y_1, y_2)$ is a solution to the linear equation:

$$y_0 = y_1y_2 \qquad (5.16)$$

However, equation is essentially quadratic already in the case of rhombus [1].

5.3 Evaluating hyperdeterminant

In general resolving eqs.(5.5) is rather tedious, moreover (5.9) provides U and W as solutions to biquadratic equations, which are of limited practical use. However, since we need $y_0(y_1, y_2)$, there is no need to find U and W: this function is defined by discriminantal condition and what we actually need is evaluation of hyperdeterminant. This is a straightforward calculation with a nice answer:

$$D_{2|3} \sim \Big\{\mathbf{P}_+\mathbf{Q}_+\mathbf{Q}_-\Big\}\Big\{\mathbf{P}_-\mathbf{Q}_+\mathbf{Q}_-\Big\} - \Big\{\mathbf{P}_+\mathbf{P}_-\mathbf{Q}_+\Big\}\Big\{\mathbf{P}_+\mathbf{P}_-\mathbf{Q}_-\Big\} \tag{5.17}$$

where $\Big\{\mathbf{PQR}\Big\} \equiv \epsilon^{\lambda\mu\nu}P^\lambda Q^\mu R^\nu$ is the mixed product of three 3-component vectors. Proportionality coefficient between the first and the second lines in (5.17) is -1 for Minkovski signature. Vectors $\mathbf{P}_\pm$ and $\mathbf{Q}_\pm$ are still another version of parametrization of (5.5):

$$U\mathbf{P}_+ \ + U^{-1}\mathbf{P}_- \ + W\mathbf{Q}_+ \ + W^{-1}\mathbf{Q}_- = 0 \tag{5.18}$$

i.e.

$$\mathbf{P}_+ = z_1(\mathbf{y} - \mathbf{y}_A), \qquad \mathbf{P}_- = z_1(\mathbf{y} - \mathbf{y}_C), \qquad \mathbf{Q}_+ = z_2(\mathbf{y} - \mathbf{y}_B), \qquad \mathbf{Q}_- = z_2(\mathbf{y} - \mathbf{y}_D) \tag{5.19}$$

Note that $D_{2|3}$ itself is of the 16-th power in components of $\mathbf{P}$ and $\mathbf{Q}$, moreover it depends on particular choice of a pair of equations out of nine in (5.9). However, all these 36 versions of $D_{2|3}$ contain one and the same factor (5.17), which is the quadratic equation for y_0 that we are looking for. Quadraticity is obvious in the first line of (5.17) and is obscure in representation through scalar products, which is still also useful in applications.

5.4 Examples

Eq.(5.17) provides $y_0(y_1, y_2)$ for generic quadrilateral as a function of positions of its four vertices in $\mathbf{y}$-space. According to (5.2) these $4 \times 3 = 12$ components of $\mathbf{y}_a = \frac{\mathbf{v}_a}{z_a}$ are not *free* parameters (i.e. can not be chosen in arbitrary way): they are expressed through $3 \times 2 - 1 = 5$ components of the three independent null-vectors, constrained by the inscribed circle condition $l_1 + l_3 = l_2 + l_4$. Two of these five free parameters depend on the choice of the general orientation and scale, so that finally the whole pattern of boundary conditions is labeled by 3 parameters and they can be chosen in different ways.

Mixed products with $\mathbf{P}$ and $\mathbf{Q}$ from (5.19) are actually all *linear* in $\mathbf{y}$:

$$\begin{array}{llcl} ABD: & \Big\{\mathbf{P}_+\mathbf{Q}_+\mathbf{Q}_-\Big\} & = & z_1z_2^2\Big(\mathbf{y}\cdot\big([\mathbf{y}_A\times\mathbf{y}_B]+[\mathbf{y}_B\times\mathbf{y}_D]+[\mathbf{y}_D\times\mathbf{y}_A]\big) \ - \ \{\mathbf{y}_A\mathbf{y}_B\mathbf{y}_D\}\Big) \\ CBD: & \Big\{\mathbf{P}_-\mathbf{Q}_+\mathbf{Q}_-\Big\} & = & z_1z_2^2\Big(\mathbf{y}\cdot\big([\mathbf{y}_C\times\mathbf{y}_B]+[\mathbf{y}_B\times\mathbf{y}_D]+[\mathbf{y}_D\times\mathbf{y}_C]\big) \ + \ \{\mathbf{y}_B\mathbf{y}_C\mathbf{y}_D\}\Big) \\ ACB: & \Big\{\mathbf{P}_+\mathbf{P}_-\mathbf{Q}_+\Big\} & = & z_1^2z_2\Big(\mathbf{y}\cdot\big([\mathbf{y}_A\times\mathbf{y}_C]+[\mathbf{y}_C\times\mathbf{y}_B]+[\mathbf{y}_B\times\mathbf{y}_A]\big) \ + \ \{\mathbf{y}_A\mathbf{y}_B\mathbf{y}_C\}\Big) \\ ACD: & \Big\{\mathbf{P}_+\mathbf{P}_-\mathbf{Q}_-\Big\} & = & z_1^2z_2\Big(\mathbf{y}\cdot\big([\mathbf{y}_A\times\mathbf{y}_C]+[\mathbf{y}_D\times\mathbf{y}_A]+[\mathbf{y}_C\times\mathbf{y}_D]\big) \ - \ \{\mathbf{y}_A\mathbf{y}_C\mathbf{y}_D\}\Big) \end{array} \tag{5.20}$$

Each line in (5.20) can also be written as a sum of four 3×3 determinants, for example,

$$ABD: \quad z_1z_2^2\left(\left\|\begin{array}{ccc} y_0 & y_1 & y_2 \\ y_{A0} & y_{A1} & y_{A2} \\ y_{B0} & y_{B1} & y_{B2}\end{array}\right\| + \left\|\begin{array}{ccc} y_0 & y_1 & y_2 \\ y_{B0} & y_{B1} & y_{B2} \\ y_{D0} & y_{D1} & y_{D2}\end{array}\right\| + \left\|\begin{array}{ccc} y_0 & y_1 & y_2 \\ y_{D0} & y_{D1} & y_{D2} \\ y_{A0} & y_{A1} & y_{A2}\end{array}\right\| - \left\|\begin{array}{ccc} y_{A0} & y_{A1} & y_{A2} \\ y_{B0} & y_{B1} & y_{B2} \\ y_{D0} & y_{D1} & y_{D2}\end{array}\right\|\right)$$

It remains to substitute particular values of $\mathbf{y}_a$ and z_a in order to obtain concrete equations in concrete examples.

5.4.1 Square

From Fig.9 and (5.13),

$$\mathbf{y}_A = (-1; 1, -1), \quad \mathbf{y}_B = (1; 1, 1), \quad \mathbf{y}_C = (-1; -1, 1), \quad \mathbf{y}_D = (1; -1, -1) \tag{5.21}$$

Substituting these vectors for lines in determinants, we obtain for the first line in (5.20):

$$ABD: \quad z_1z_2^2\left(\left\|\begin{array}{ccc} y_0 & y_1 & y_2 \\ -1 & 1 & -1 \\ 1 & 1 & 1\end{array}\right\| + \left\|\begin{array}{ccc} y_0 & y_1 & y_2 \\ 1 & 1 & 1 \\ 1 & -1 & -1\end{array}\right\| + \left\|\begin{array}{ccc} y_0 & y_1 & y_2 \\ 1 & -1 & -1 \\ -1 & 1 & -1\end{array}\right\| - \left\|\begin{array}{ccc} -1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -1\end{array}\right\|\right) =$$

$$= 4z_1z_2^2(y_0 + y_1 - y_2 - 1) \tag{5.22}$$

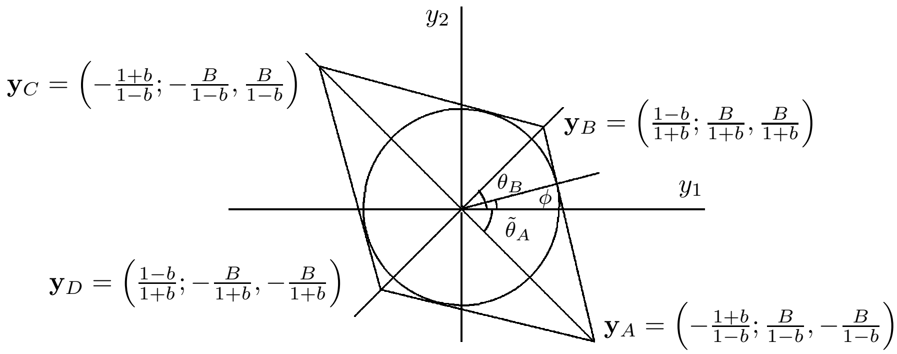

Figure 11: Rhombus in the standard parametrization, suggested in [4]. The values of y_0 are also shown, $y_0 = 0$ at four tangent points. The angle ϕ defines the direction of a normal to the rhombus side. Directions to the vertices are $\theta_A = -\frac{\pi}{4}$ (so that $\tilde\theta_A = 2\pi - \theta_A = \frac{\pi}{4}$), $\theta_B = \frac{\pi}{4}$, $\theta_C = \frac{3\pi}{4}$, $\theta_D = \frac{5\pi}{4}$. Parameter $B = \sqrt{1+b^2}$. External momenta $\mathbf{p}_a = \mathbf{y}_{a+1} - \mathbf{y}_a$ are vectors along the sides, i.e. are given by differences between the values that $\mathbf{y}$ takes at vertices. Parameters z_a are made from scalar products of these vectors and therefore are derived from the data in the picture.

Similarly for the other three lines we get:

$$CBD: \quad z_1 z_2^2 \left(\begin{Vmatrix} y_0 & y_1 & y_2 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{Vmatrix} + \begin{Vmatrix} y_0 & y_1 & y_2 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{Vmatrix} + \begin{Vmatrix} y_0 & y_1 & y_2 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{Vmatrix} + \begin{Vmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \end{Vmatrix} \right) =$$

$$= 4 z_1 z_2^2 (-y_0 + y_1 - y_2 + 1) \tag{5.23}$$

$$ACB: \quad z_1^2 z_2 \left(\begin{Vmatrix} y_0 & y_1 & y_2 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{Vmatrix} + \begin{Vmatrix} y_0 & y_1 & y_2 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{Vmatrix} + \begin{Vmatrix} y_0 & y_1 & y_2 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{Vmatrix} + \begin{Vmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & -1 & 1 \end{Vmatrix} \right) =$$

$$= 4 z_1^2 z_2 (-y_0 + y_1 + y_2 - 1) \tag{5.24}$$

$$ACD: \quad z_1^2 z_2 \left(\begin{Vmatrix} y_0 & y_1 & y_2 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{Vmatrix} + \begin{Vmatrix} y_0 & y_1 & y_2 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \end{Vmatrix} + \begin{Vmatrix} y_0 & y_1 & y_2 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \end{Vmatrix} - \begin{Vmatrix} -1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \end{Vmatrix} \right) =$$

$$= 4 z_1^2 z_2 (y_0 + y_1 + y_2 + 1) \tag{5.25}$$

Since in this case $z_1 = z_2 = \frac{1}{4}$ we finally obtain for (5.17) the familiar result (5.16):

$$\mathcal{S}_\square \sim D_{2|3} = \frac{1}{16^2}\Big((y_0 + y_1 - y_2 - 1)(-y_0 + y_1 - y_2 + 1) - (-y_0 + y_1 + y_2 - 1)(y_0 + y_1 + y_2 + 1) \Big) =$$

$$= \frac{1}{16^2}\Big((y_1 - y_2)^2 - (y_0 - 1)^2 - (y_1 + y_2)^2 + (y_0 + 1)^2 \Big) = \frac{1}{64}(y_0 - y_1 y_2) = 0 \tag{5.26}$$

5.4.2 Rhombus

According to the table in s.2.6.3 of [1], see also Fig.11,

$$\mathbf{y}_A = (-b_-, -B_-, B_-), \quad \mathbf{y}_B = (B_+; B_+, b_+), \quad \mathbf{y}_C = (-b_-; B_-, -B_-), \quad \mathbf{y}_D = (b_+; -B_+, -B_+), \tag{5.27}$$

where

$$b_- = \frac{1+b}{1-b}, \quad b_+ = \frac{1-b}{1+b}, \quad B_- = \frac{B}{1-b}, \quad B_+ = \frac{B}{1+b}, \quad B = \sqrt{1+b^2} \tag{5.28}$$

The four lines in (5.20) are now

$$z_1 z_2^2 \left(\begin{Vmatrix} y_0 & y_1 & y_2 \\ -b_- & B_- & -B_- \\ b_+ & B_+ & B_+ \end{Vmatrix} + \begin{Vmatrix} y_0 & y_1 & y_2 \\ b_+ & B_+ & B_+ \\ b_+ & -B_+ & -B_+ \end{Vmatrix} + \begin{Vmatrix} y_0 & y_1 & y_2 \\ b_+ & -B_+ & -B_+ \\ -b_- & B_- & -B_- \end{Vmatrix} - \begin{Vmatrix} -b_- & B_- & -B_- \\ b_+ & B_+ & B_+ \\ b_+ & -B_+ & -B_+ \end{Vmatrix} \right) =$$

$$= 4z_1z_2^2B_+\left(B_-(y_0-b_+)+\frac{1}{2}(b_++b_-)(y_1-y_2)\right) \tag{5.29}$$

$$z_1z_2^2\left(\left\|\begin{array}{ccc} y_0 & y_1 & y_2\\ -b_- & -B_- & B_-\\ b_+ & B_+ & B_+\end{array}\right\|+\left\|\begin{array}{ccc} y_0 & y_1 & y_2\\ b_+ & B_+ & B_+\\ b_+ & -B_+ & -B_+\end{array}\right\|+\left\|\begin{array}{ccc} y_0 & y_1 & y_2\\ b_+ & -B_+ & -B_+\\ -b_- & -B_- & B_-\end{array}\right\|+\left\|\begin{array}{ccc} b_+ & B_+ & B_+\\ -b_- & -B_- & B_-\\ b_+ & -B_+ & -B_+\end{array}\right\|\right)=$$

$$= 4z_1z_2^2B_+\left(-B_-(y_0-b_+)+\frac{1}{2}(b_++b_-)(y_1-y_2)\right) \tag{5.30}$$

$$z_1^2z_2\left(\left\|\begin{array}{ccc} y_0 & y_1 & y_2\\ -b_- & B_- & -B_-\\ -b_- & -B_- & B_-\end{array}\right\|+\left\|\begin{array}{ccc} y_0 & y_1 & y_2\\ -b_- & -B_- & B_-\\ b_+ & B_+ & B_+\end{array}\right\|+\left\|\begin{array}{ccc} y_0 & y_1 & y_2\\ b_+ & B_+ & B_+\\ -b_- & B_- & -B_-\end{array}\right\|+\left\|\begin{array}{ccc} -b_- & B_- & -B_-\\ b_+ & B_+ & B_+\\ -b_- & -B_- & B_-\end{array}\right\|\right)=$$

$$= 4z_1^2z_2B_-\left(-B_+(y_0+b_-)+\frac{1}{2}(b_++b_-)(y_1+y_2)\right) \tag{5.31}$$

$$z_1^2z_2\left(\left\|\begin{array}{ccc} y_0 & y_1 & y_2\\ -b_- & B_- & -B_-\\ -b_- & -B_- & B_-\end{array}\right\|+\left\|\begin{array}{ccc} y_0 & y_1 & y_2\\ b_+ & -B_+ & -B_+\\ -b_- & B_- & -B_-\end{array}\right\|+\left\|\begin{array}{ccc} y_0 & y_1 & y_2\\ -b_- & -B_- & B_-\\ b_+ & -B_+ & -B_+\end{array}\right\|-\left\|\begin{array}{ccc} -b_- & B_- & -B_-\\ -b_- & -B_- & B_-\\ b_+ & -B_+ & -B_+\end{array}\right\|\right)=$$

$$= 4z_1^2z_2B_-\left(B_+(y_0+b_-)+\frac{1}{2}(b_++b_-)(y_1+y_2)\right) \tag{5.32}$$

Therefore we obtain for (5.17):

$$(4z_1z_2)^2\left\{(z_2B_+)^2\left(\frac{1}{4}(b_++b_-)^2(y_1-y_2)^2-B_-^2(y_0-b_+)^2\right)-(z_1B_-)^2\left(\frac{1}{4}(b_++b_-)^2(y_1+y_2)^2-B_+^2(y_0+b_-)^2\right)\right\}=$$

$$=(4z_1z_2)^2\Big\{(B_+B_-)^2\Big((z_1^2-z_2^2)y_0^2+2(z_1^2b_-+z_2^2b_+)y_0+(z_1b_-)^2-(z_2b_+)^2\Big)+$$

$$+\frac{1}{4}(b_++b_-)^2\left[\left((z_2B_+)^2-(z_1B_-)^2\right)(y_1^2+y_2^2)-2\left((z_2B_+)^2+(z_1B_-)^2\right)y_1y_2\right]\Big\}\overset{(5.28)}{=}$$

$$=\left(\frac{2z_1z_2B^2}{1-b^2}\right)^2\Bigg\{(z_1^2-z_2^2)y_0^2+2y_0\frac{z_1^2(1+b)^2+z_2^2(1-b)^2}{1-b^2}+\left(\frac{z_1(1+b)}{1-b}\right)^2-\left(\frac{z_2(1-b)}{1+b}\right)^2+$$

$$+B^2\left[\left(\frac{z_2}{1+b}\right)^2-\left(\frac{z_1}{1-b}\right)^2\right](y_1^2+y_2^2)-2B^2\left[\left(\frac{z_2}{1+b}\right)^2+\left(\frac{z_1}{1-b}\right)^2\right]y_1y_2\Bigg\}=$$

$$=\left(\frac{8z_1z_2zB^2}{1-b^2}\right)^2\Big((1-b^2)y_0+b(1-y_0^2)-(1+b^2)y_1y_2\Big) \tag{5.33}$$

provided

$$z_1=(1-b)z,\quad z_2=(1+b)z, \tag{5.34}$$

what is indeed the case for rhombus, with

$$z=\frac{1}{4\sqrt{1+b^2}}, \tag{5.35}$$

see [4, 13].

Thus we see that exact solution to NG equations with rhombus in the role of the boundary Π is

$$\mathcal{S}_\diamond = y_1y_2-\frac{1}{2}(1-y_0^2)\sin(2\phi)-y_0\cos(2\phi)=0 \tag{5.36}$$

where

$$\sin(2\phi)=\frac{2b}{1+b^2},\quad \cos(2\phi)=\frac{1-b^2}{1+b^2} \tag{5.37}$$

This is in accordance with eq.(2.54) of [1].

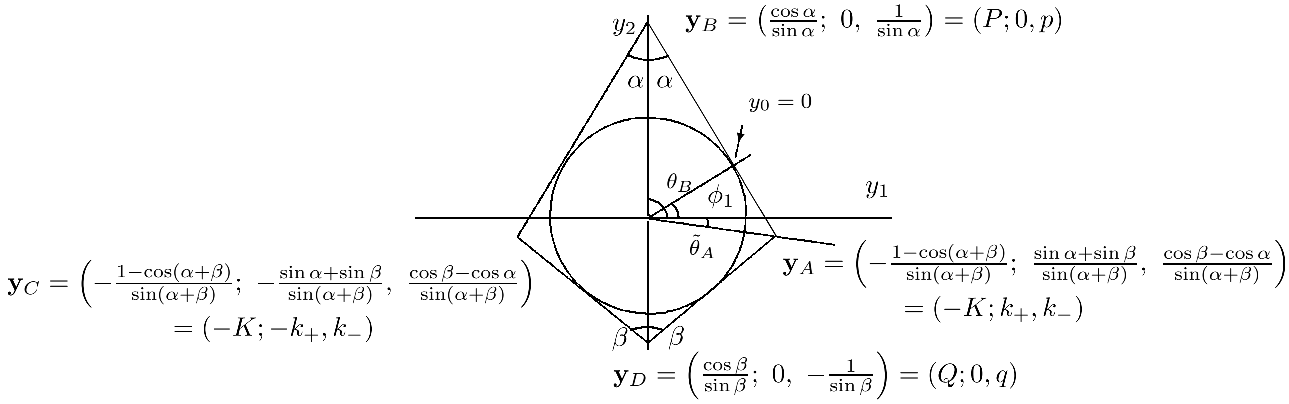

Figure 12: Kite-like polygon $\bar\Pi$ with only one Z_2-symmetry, $y_1 \to -y_1$. Kites form a two-dimensional family, parameterized by α and β. Angles at four vertices are $\pi-\alpha-\beta$ at A and C, 2α at B and 2β at D and directions to vertices are $\theta_A = 2\pi - \tilde\theta_A = \frac{\alpha-\beta}{2}$, $\theta_B = \frac{\pi}{2}$, $\theta_C = \pi - \frac{\alpha-\beta}{2}$ and $\theta_D = \frac{3\pi}{2}$. The four normal directions are: $\phi_1 = \alpha$, $\phi_2 = \pi-\phi_1 = \pi-\alpha$, $\phi_3 = \pi+\beta$ and $\phi_4 = 2\pi-\beta$. Rhombus is a particular sub-family with $\beta=\alpha$. Note that this picture is rotated by an angle $\frac{\pi}{4}$ as compared to Fig.11.

5.4.3 Kite

Kites form a two-dimensional family of polygons $\bar\Pi$, which possess only one Z_2-symmetry, $y_1 \leftrightarrow -y_1$. We parameterize them by two angle variables α and β, which are *halves* of the angles at two non-equivalent vertices, see Fig.12. Rhombi with symmetry, enhanced to $Z_2\times Z_2$, $y_2 \leftrightarrow -y_2$ in addition to $y_1 \leftrightarrow -y_1$ are a one-parametric sub-family of kites with $\alpha=\beta$. Note that for comparison with the results of s.5.4.2 one should also make a rotation of the (y_1,y_2) plane by $\frac{\pi}{4}$. After this rotation the square solution (5.16) turns into

$$\mathcal{S}'_\square = 2y_0 + y_1^2 - y_2^2 = 0 \tag{5.38}$$

and rhombic solution (5.36) – into

$$\mathcal{S}'_\diamond = 2y_0\cos(2\phi) + (1-y_0^2)\sin(2\phi) + y_1^2 - y_2^2 = 0 \tag{5.39}$$

or

$$\mathcal{S}'_\diamond \sim 2(1-b^2)y_0 + 2b(1-y_0^2) + (1+b^2)(y_1^2-y_2^2) = 0 \tag{5.40}$$

with $\phi = \frac{\pi}{4} - \alpha$ and

$$b = \frac{|\cos\alpha - \sin\alpha|}{\cos\alpha+\sin\alpha} \tag{5.41}$$

It is a simple geometrical exercise to express the values of $\mathbf{y}$ at the kite vertices through α and β. It is only important to remember that we put the radius of inscribed circle equal to one. It follows that the ordinates of the vertices B and D are $y_{2B} = \cot\alpha$ and $y_{2D} = \cot\beta$, while the corresponding values of y_0 are $y_{0B} = \frac{1}{\sin\alpha}$ and $y_{0D} = \frac{1}{\sin\beta}$, because y_0 vanishes at the tangent points with the unit circle. Further, the two side lengths $l_1 = l_{AB}$ and $l_4 = l_{DA}$ of the kite are related through

$$\begin{aligned} l_1\cos\alpha + l_4\cos\beta &= \frac{1}{\sin\alpha} + \frac{1}{\sin\beta}, \\ l_1\sin\alpha = l_4\sin\beta &= y_{1A} \end{aligned} \tag{5.42}$$

The most convenient variables for actual calculations are $t = \tan\frac{\alpha}{2}$ and $t' = \tan\frac{\beta}{2}$, i.e. trigonometric functions of the *quarters* of the kite's angles with values bound between 0 and 1: $0 < t, t' < 1$. Unfortunately, they are much less convenient for consideration of particular degenerations, in particular for the square $t' = t = \tan\frac{\pi}{8} = \sqrt{\frac{\sqrt2-1}{\sqrt2+1}}$. In terms of these variables

$$\sin\alpha = \frac{2t}{1+t^2}, \quad \cos\alpha = \frac{1-t^2}{1+t^2}, \quad \sin\beta = \frac{2t'}{1+t'^2}, \quad \cos\beta = \frac{1-t'^2}{1+t'^2} \tag{5.43}$$

and

$$\begin{aligned}
\mathbf{y}_A &= \Big(-\tfrac{1-\cos(\alpha+\beta)}{\sin(\alpha+\beta)};\ \tfrac{\sin\alpha+\sin\beta}{\sin(\alpha+\beta)},\ \tfrac{\cos\beta-\cos\alpha}{\sin(\alpha+\beta)}\Big) &&= \big(-K;\ k_+,\ k_-\big) &&= \Big(-\tfrac{t+t'}{1-tt'};\ \tfrac{1+tt'}{1-tt'},\ \tfrac{t-t'}{1-tt'}\Big),\\
\mathbf{y}_B &= \big(\tfrac{\cos\alpha}{\sin\alpha};\ 0,\ \tfrac{1}{\sin\alpha}\big) &&= \big(P;\ 0,\ p\big) &&= \Big(\tfrac{1-t^2}{2t};\ 0,\ \tfrac{1+t^2}{2t}\Big),\\
\mathbf{y}_C &= \Big(-\tfrac{1-\cos(\alpha+\beta)}{\sin(\alpha+\beta)};\ -\tfrac{\sin\alpha+\sin\beta}{\sin(\alpha+\beta)},\ \tfrac{\cos\beta-\cos\alpha}{\sin(\alpha+\beta)}\Big) &&= \big(-K;\ -k_+,\ k_-\big) &&= \Big(-\tfrac{t+t'}{1-tt'};\ -\tfrac{1+tt'}{1-tt'},\ \tfrac{t-t'}{1-tt'}\Big),\\
\mathbf{y}_D &= \Big(\tfrac{\cos\beta}{\sin\beta};\ 0,\ -\tfrac{1}{\sin\beta}\Big) &&= \big(Q;\ 0,\ q\big) &&= \Big(\tfrac{1-t'^2}{2t'};\ 0,\ -\tfrac{1+t'^2}{2t'}\Big)
\end{aligned} \tag{5.44}$$

It follows that

$$\begin{aligned}
z_1 = z_A = z_C &= \frac{1}{2}\Big(\mathbf{y}_C-\mathbf{y}_B)(\mathbf{y}_B-\mathbf{y}_A)\Big)^{-1/2} = \frac{1}{2\sqrt{2}}\frac{1-tt'}{1+tt'},\\
z_2 = z_B = z_D &= \frac{1}{2}\Big(\mathbf{y}_C-\mathbf{y}_B)(\mathbf{y}_D-\mathbf{y}_C)\Big)^{-1/2} = \frac{\sqrt{2tt'}}{2(1+tt')}
\end{aligned} \tag{5.45}$$

In terms of condensed notation, introduced in (5.44), the four lines in (5.20) are now:

$$ABD: \quad z_1z_2^2\left(\left\|\begin{matrix} y_0 & y_1 & y_2\\ -K & k_+ & k_-\\ P & 0 & p\end{matrix}\right\| + \left\|\begin{matrix} y_0 & y_1 & y_2\\ P & 0 & p\\ Q & 0 & q\end{matrix}\right\| + \left\|\begin{matrix} y_0 & y_1 & y_2\\ Q & 0 & q\\ -K & k_+ & k_-\end{matrix}\right\| - \left\|\begin{matrix} -K & k_+ & k_-\\ P & 0 & p\\ Q & 0 & q\end{matrix}\right\|\right) =$$

$$= z_1z_2^2\left\{k_+\Big((p-q)y_0-(P-Q)y_2+(Pq-Qp)\Big)+\Big(K(p-q)+(P-Q)k_- -(Pq-Qp)\Big)y_1\right\}$$

$$CBD: \quad z_1z_2^2\left(\left\|\begin{matrix} y_0 & y_1 & y_2\\ -K & -k_+ & k_-\\ P & 0 & p\end{matrix}\right\| + \left\|\begin{matrix} y_0 & y_1 & y_2\\ P & 0 & p\\ Q & 0 & q\end{matrix}\right\| + \left\|\begin{matrix} y_0 & y_1 & y_2\\ Q & 0 & q\\ -K & -k_+ & k_-\end{matrix}\right\| + \left\|\begin{matrix} P & 0 & p\\ -K & -k_+ & k_-\\ Q & 0 & q\end{matrix}\right\|\right) =$$

$$= z_1z_2^2\left\{-k_+\Big((p-q)y_0-(P-Q)y_2+(Pq-Qp)\Big)+\Big(K(p-q)+(P-Q)k_- -(Pq-Qp)\Big)y_1\right\}$$

$$ACB: \quad z_1^2z_2\left(\left\|\begin{matrix} y_0 & y_1 & y_2\\ -K & k_+ & k_-\\ -K & -k_+ & k_-\end{matrix}\right\| + \left\|\begin{matrix} y_0 & y_1 & y_2\\ -K & -k_+ & k_-\\ P & 0 & p\end{matrix}\right\| + \left\|\begin{matrix} y_0 & y_1 & y_2\\ P & 0 & p\\ -K & k_+ & k_-\end{matrix}\right\| + \left\|\begin{matrix} -K & k_+ & k_-\\ P & 0 & p\\ -K & -k_+ & k_-\end{matrix}\right\|\right) =$$

$$= 2k_+z_1^2z_2\Big((k_- -p)y_0+(K+P)y_2-(Kp+Pk_-)\Big)$$

$$ACD: \quad z_1^2z_2\left(\left\|\begin{matrix} y_0 & y_1 & y_2\\ -K & k_+ & k_-\\ -K & -k_+ & k_-\end{matrix}\right\| + \left\|\begin{matrix} y_0 & y_1 & y_2\\ Q & 0 & q\\ -K & k_+ & k_-\end{matrix}\right\| + \left\|\begin{matrix} y_0 & y_1 & y_2\\ -K & -k_+ & k_-\\ Q & 0 & q\end{matrix}\right\| - \left\|\begin{matrix} -K & k_+ & k_-\\ -K & -k_+ & k_-\\ Q & 0 & q\end{matrix}\right\|\right) =$$

$$= 2k_+z_1^2z_2\Big((k_- -q)y_0+(K+Q)y_2-(Kq+Qk_-)\Big)$$

Thus (5.17) becomes:

$$\begin{aligned}
(z_1z_2)^2\Big\{ & z_2^2\Big(K(p-q)+(P-Q)k_- -(Pq-Qp)\Big)^2y_1^2-(k_+z_2)^2\Big((p-q)y_0-(P-Q)y_2+(Pq-Qp)\Big)^2-\\
& -(2k_+z_1)^2\Big((k_- -p)y_0+(K+P)y_2-(Kp+Pk_-)\Big)\Big((k_- -q)y_0+(K+Q)y_2-(Kq+Qk_-)\Big)\Big\}
\end{aligned}$$

and finally

$$\begin{aligned}
\mathcal{S}_{kite} = D_{2|3} = & \frac{1}{128(1+tt')^2}\Big\{\Big((1-tt')^2-2(t^2+t'^2)\Big)y_0^2-4(t-t')(t+t')y_0y_2+\\
& +(1+tt')^2(y_1^2-y_2^2)-2(t-t')^2y_2^2+4(1-tt')\Big((t+t')y_0+(t-t')y_2\Big)-\big(1-6tt'+(tt')^2\big)\Big\}
\end{aligned} \tag{5.46}$$

This expression can be also rewritten as

$$64\mathcal{S}_{kite} = \frac{1}{2}(y_1^2+y_2^2-y_0^2-1)+\frac{(1-t^2)(1-t'^2)}{(1+tt')^2}y_0^2-\frac{(1+t^2)(1+t'^2)}{(1+tt')^2}y_2^2+\frac{4tt'}{(1+tt')^2}-$$

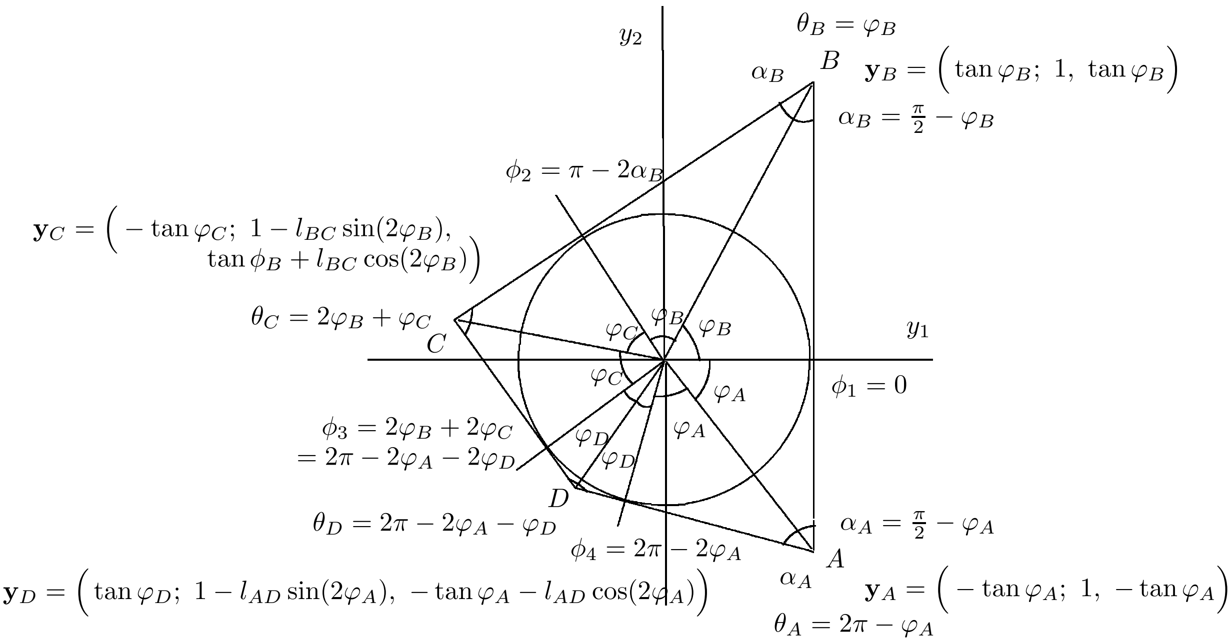

Figure 13: Generic skew quadrilateral $\bar{\Pi}$ is parameterized by four angles: $\varphi_{A,B,C,D}$, subjected to constraint $2\varphi_A + 2\varphi_B + 2\varphi_C + 2\varphi_D = 2\pi$. For circle of unit radius the side lengths are $l_1 = l_{AB} = \tan\varphi_A + \tan\varphi_B$ etc. Rotation freedom is fixed by requiring that the first segment AB is vertical: $\phi_1 = 0$. Then the other three normal directions are: $\phi_2 = \phi_{BC} = 2\varphi_B$ $\phi_3 = \phi_{CD} = 2\varphi_B + 2\varphi_C = 2\pi - 2\varphi_A - 2\varphi_D$, $\phi_4 = \phi_{DA} = 2\pi - 2\varphi_D$ and direction towards the vertices are: $\theta_A = 2\pi - \varphi_A$ $\theta_B = \varphi_B$, $\theta_C = 2\varphi_B + \varphi_C$, $\theta_D = 2\pi - 2\varphi_A - \varphi_D$. The angles of quadrilateral are $2\alpha_A = \pi - 2\varphi_A$, $2\alpha_B = \pi - 2\varphi_B$, $2\alpha_C = \pi - 2\varphi_C$ and $2\alpha_D = \pi - 2\varphi_D$ (α_C and α_D are not shown).

$$-\frac{2(t^2 - t'^2)}{(1+tt')^2} y_0 y_2 + \frac{2(1-tt')}{(1+tt')^2}\Big((t+t')y_0 + (t-t')y_2\Big)$$

or, making use of (5.43) to convert back to original angular variables:

$$\begin{aligned}\frac{1}{2}\Big(128\mathcal{S}_{kite} + P_2\Big)\cos(\alpha-\beta) =& \\ = y_0^2\cos\alpha\cos\beta - y_2^2 + y_0y_2(\cos\alpha - \cos\beta) + y_0\sin(\alpha+\beta) + y_2(\sin\alpha - \sin\beta) + \sin\alpha\sin\beta =& \\ = y_0^2\cos\alpha\cos\beta + y_0y_2(\cos\alpha-\cos\beta) + y_0\sin(\alpha+\beta) + (\sin\alpha - y_2)(\sin\beta + y_2)&\end{aligned} \tag{5.47}$$

5.4.4 A version of parametrization for generic quadrilateral case

In the case of generic quadrilateral (with inscribed circle) we have, see Fig.13:

$$l_{AB} = \tan\varphi_A + \tan\varphi_B, \quad \mathbf{y}_B - \mathbf{y}_A = \Big(\sigma_{AB}l_{AB};\ -l_{AB}\sin\phi_{AB},\ l_{AB}\cos\phi_{AB}\Big) \tag{5.48}$$

and similarly for all other sides. Therefore, assuming that the first side AB is parallel to ordinate axis, we can parameterize all vertices by four angles φ_a constrained by a single relation:

$$\varphi_A + \varphi_B + \varphi_C + \varphi_D = \pi \tag{5.49}$$

Then

$$\begin{aligned}\mathbf{y}_A &= \Big(-\tan\varphi_A;\ 1,\ -\tan\varphi_A\Big),\\ \mathbf{y}_B &= \Big(\tan\varphi_B;\ 1,\ \tan\varphi_B\Big),\\ \mathbf{y}_C &= \Big(-\tan\varphi_C;\ 1 - l_{BC}\sin(2\varphi_B),\ \tan\phi_B + l_{BC}\cos(2\varphi_B)\Big),\\ \mathbf{y}_D &= \Big(\tan\varphi_D;\ 1 - l_{AD}\sin(2\varphi_A),\ -\tan\varphi_A - l_{AD}\cos(2\varphi_A)\Big)\end{aligned} \tag{5.50}$$

One should further substitute

$$\cos(2\varphi) = \frac{1-\tan^2\varphi}{1+\tan^2\varphi}, \quad \sin(2\varphi) = \frac{2\tan\varphi}{1+\tan^2\varphi}, \tag{5.51}$$

then the constraint (5.49) is a simple relation

$$t_A + t_B + t_C + t_D = t_At_Bt_C + t_At_Bt_D + t_At_Ct_D + t_Bt_Ct_D, \tag{5.52}$$

linear in all t-variables. If, say, t_D is expressed through the three other variables, then

$$1 + t_D^2 = \frac{(1+t_A^2)(1+t_B^2)(1+t_C^2)}{t_At_B + t_Bt_C + t_Ct_A - 1)^2} \tag{5.53}$$

and

$$z_1 = \sqrt{\frac{1+t_B^2}{8(t_A+t_B)(t_B+t_C)}}, \quad z_2 = \sqrt{\frac{t_At_B+t_Bt_C+t_Ct_A-1}{8(t_A+t_B)(t_B+t_C)}} \tag{5.54}$$

Evaluation of discriminant (5.17) is straightforward and results in:

$$\begin{aligned}
\mathcal{S}_{quadri} = D_{2|3} \sim y_0^2\Big(-t_At_B - t_Bt_C + t_At_C + (2t_At_C-1)t_B^2\Big) + \Big(2 - t_At_B - t_Bt_C - t_Ct_A + t_B^2\Big) + \\
+y_1^2\Big(2 + t_At_B - 3t_Bt_C + t_At_C - t_B^2\Big) + y_2^2\Big(-3t_At_B + t_Bt_C - t_At_C + (2t_At_C+1)\Big) + \\
+2y_1y_2\Big(-t_A + 2t_B + t_C + (t_A - t_C)t_B^2 + 2t_At_Bt_C\Big) + 2y_0y_1\Big(t_A + t_C + 2(-t_A + t_C)t_B^2 - 2t_At_Bt_C\Big) + \\
+4y_0y_2t_At_B(1-t_Bt_C) - 2y_0(t_A+t_C)(1+t_B^2) - 4y_1(1-t_Bt_C) + 2y_2\Big(t_A - 2t_B - t_C + (t_A+t_C)t_B^2\Big)
\end{aligned} \tag{5.55}$$

Omitted overall coefficient (unneeded for our purposes) is

$$\frac{(z_1z_2)^2(t_A+t_B)(t_B+t_C)}{2(1+t_B^2)(t_At_B+t_Bt_C+t_Ct_A-1)^2} \tag{5.56}$$

This is a rather long expression and it is asymmetric in its variables, because use *independent* variables, with t_D excluded. Actually this formula possesses cyclic symmetry under $(ABCD) \to (BCDA) \to \ldots$and is also invariant under permutations of *opposite* vertices $B \leftrightarrow D$ and $A \leftrightarrow C$. Only the last of these three symmetries is explicit in (5.55).

Particular case of square corresponds to $t_A = t_B = t_C = \tan\frac{\pi}{4} = 1$, then (5.55) becomes

$$\mathcal{S}_{quadri} \ \stackrel{t_A=\ldots=1}{\longrightarrow}\ -8(y_0 - y_1y_2) \sim \mathcal{S}_\Box, \tag{5.57}$$

as expected.

Comparison with the rhombus case is a little more involved. For rhombus $t_A = t_B^{-1} = t_C = t_D^{-1}$: pairs of opposite angles are equal, the sum of adjacent angles is π (this is true for any parallelogram, but inscribed circle condition leaves only rhombi for our consideration). Expressed through $t_B = t$, eq.(5.55) becomes:

$$\mathcal{S}_{quadri} \ \stackrel{t_A=t_C=t_B^{-1}}{\longrightarrow}\ -4\left(t+\frac{1}{t}\right)\left\{y_0 - y_1y_2 + \frac{1}{4}\left(t-\frac{1}{t}\right)(y_0^2+y_1^2-y_2^2-1)\right\} \sim \mathcal{S}_\diamond \tag{5.58}$$

In order to compare this expression with other versions of $\mathcal{S}_\diamond$ originated by [4], we should rotate it in the (y_1, y_2) plane to switch from the choice of vertical side AB, implied in 5.55, to $\theta_B = \frac{\pi}{4}$, implied in (5.36). This means that we should rotate by angle $\phi_1 = \phi_{AB}$, which is related to $t = t_B = \tan(\varphi_B)$ with $\varphi_B = \frac{\pi}{4} - \phi_{AB}$ by

$$\tan(2\phi) = \cot(2\varphi_B) = \frac{\cos^2\varphi_B - \sin^2\varphi_B}{2\sin\varphi_B\cos\varphi_B} = -\frac{1}{2}\left(t - \frac{1}{t}\right) \tag{5.59}$$

Substituting $(y_1, y_2) \to (y_1\cos\phi + y_2\sin\phi,\ -y_1\sin\phi + y_2\cos\phi)$ into (5.58) we convert the r.h.s. into

$$\begin{aligned}
y_0 - y_1y_2\cos(2\phi) + \frac{1}{2}(y_1^2 - y_2^2)\sin(2\phi) - \frac{1}{2}\tan(2\phi)\Big(y_0^2 + 2y_1y_2\sin(2\phi) + (y_1^2-y_2^2)\cos(2\phi) - 1\Big) = \\
= \frac{1}{\cos(2\phi)}\left(y_0\cos(2\phi) - y_1y_2 + \frac{1}{2}(1-y_0^2)\sin(2\phi)\right) \stackrel{5.36)}{=} -\frac{1}{\cos(2\phi)}\mathcal{S}_\diamond
\end{aligned} \tag{5.60}$$

One more way to represent $\mathcal{S}_{quadri}$ is to express it through canonical elements P_2 and $\mathcal{L}_{quadri}$, which is linear in y_0 with coefficient one:

$$\mathcal{S}_{quadri} \sim \mathcal{L}_{quadri} + \mu_{quadri} P_2 \tag{5.61}$$

From (5.55)

$$\mu_{quadri} = -\frac{t_A t_C - (t_A + t_C)t_B + (2t_A t_C - 1)t_B^2}{2(t_A + t_C)(1 + t_B^2)} \tag{5.62}$$

It turns into $\mu_\square = 0$ for the square (when all four $t_a = 1$) and into $\mu_\diamond = \frac{1}{2}\left(t - \frac{1}{t}\right) = -\frac{1}{2}\tan(2\phi_{AB})$ for rhombus.

5.5 Intermediate conclusion

The main result of this section is that *exact solution* to our Plateau problem for generic skew quadrilateral Π is reduced to *quadratic* equation in y-variables:

$$\mathcal{S}_\Pi(y_0; y_1, y_2) = 0 \tag{5.63}$$

Moreover, it is quadratic in y_0. Only in the case of the square, $\bar{\Pi} = \square$, i.e. for Z_4-symmetric configuration, it further reduces to a linear (5.16). This means that *such* elements, more sophisticated than linear, but still simple, should be of primary interest for us in the study of the boundary ring at least at $n = 4$. This new experience implies certain modification of research direction, suggested in sections 2.5 and 2.6 on the base of Z_n-symmetric considerations, shifting attention from y_0-linearity of the desired boundary ring elements.

In the next section 6 we continue discussion of the boundary ring structure, originated in [1]. Not-surprisingly, $\mathcal{S}_\Pi$ is not immediately distinguished as an element of $\mathcal{R}_\Pi$ – it belongs to the intersection of the ring with the space of NG solutions and can not be found by considerations of the ring only, – but it can be easily found within the *simple* classes of elements in $\mathcal{R}_\Pi$. A systematic approach can be to classify the elements of $\mathcal{R}_\Pi$ of a given degree in y-variables, and then use them as anzatze for solutions to Plateau problem. Such anzatze will contain a few free parameters ("moduli"), because degree does not fix the element of $\mathcal{R}_\Pi$ unambiguously. They can be either perturbed, substituted into NG equations and analyzed by the methods of s.comprec or, instead, as suggested in [13], used to evaluate the regularized action, which can be afterwards minimized w.r.t. the remaining "moduli". This provides two approximate methods, which can occasionally produce exact answers (and then coincide). It would be particularly interesting to analyze in detail the families $\mathcal{F}_{n/2}$ of degree $n/2$ in $\mathcal{R}_\Pi$. Not only exact solutions $\mathcal{S}_\Pi$ at $n = 2$ and $n = 4$ belong to $\mathcal{F}_{n/2}$, such families looks distinguished in the theory of the rings themselves: $n/2$ looks like the lowest degree necessary to distinguish between the ring itself and its sub-rings, associated with unifications of Π with additional lines.

6 Boundary ring for polygons

This section is devoted to simple arithmetics of the polygon boundary ring and is a first step towards their systematic consideration on the lines, implied by s.5.5. Essential simplification of $\mathcal{R}_\Pi$ is provided by conditions (1.2) and we continue to impose them. Then $P_2 = y_0^2 + 1 - y_1^2 - y_2^2 = y_0^2 + 1 - z\bar z$ is always an element of $\mathcal{R}_\Pi$, but we need more. The situation is not quite simple because generically there are no "generators" in the rings of polynomials of many variables,[6] instead a sophisticated structure arises of complementary maximal ideals and "dual" descriptions. We do not go in details of abstract algebra in this paper[7] and concentrate on the down-to-earth consideration of low-degree elements in $\mathcal{R}_\Pi$, to provide concrete information for further developments. Our "universal" P_2 is of degree two, but the other "obvious" polynomials P_Π, considered in [1] and listed in (2.6), are of the "high" degree n. At the same time, $K_{n/2}$ in (2.16) and $\mathcal{S}$ in s.5.4 are of degree $n/2$ and still belong to $\mathcal{R}_\Pi$.

In order to put the situation under control we fully use the specifics of our boundary ring: the fact that Π consists of intersecting straight segments (actually, entire lines, if we are interested in polynomial boundary rings) and thus can be constructed from elementary rings for individual straight lines. This allows to introduce complex-valued elements $\mathcal{C}_\Pi \in \mathcal{R}_\Pi$, which, like P_Π, are multiplicative characters, i.e. are products of the elementary $\mathcal{C}_|$ for individual segments. Of course, they are also of degree n in y-variables. Then we demonstrate how the relevant real-valued elements of lower degree can be extracted in a generalizable fashion.

6.1 A single null segment

According to (2.8),

$$z = y_1 + iy_2 = e^{i\phi}\Big(h + i\sigma(y_0 - \mathrm{y}_{00})\Big), \tag{6.1}$$

where, see Fig.14, ϕ is an angle between a normal to the segment and the y_1-axis, h is the length of the normal from the origin to its intersection point with the straight line which contains our segment, y_{00} is the value of y_0 at this intersection point, while $\sigma = \pm 1$, depending on the direction of y_0. This relation defines an element of the boundary ring,

$$\mathcal{C}_| = \mathcal{C}_|(\phi,\sigma|h,\mathrm{y}_{00}) = y_0 - \mathrm{y}_{00} - i\sigma\left(h - e^{-i\phi}z\right) \tag{6.2}$$

which vanishes along the segment. In fact it vanishes on entire straight line, which contains the segment. Of course, this property is inherited by boundary rings in all more complicated situations: polynomials vanishing on the sides of a polygon will do so on entire straight lines, containing these segments. This is a general feature of any approach based on polynomials, though it is not necessarily preserved in transition from algebraic geometry to functional analysis. It deserves mentioning that solutions to Plateau problem in flat Euclidean space are believed to respect this property, see, for example, [33].

Actually the real and imaginary parts of (6.2) are the two independent generators of $\mathcal{R}_{\rm segment}$:

$$\begin{aligned}\mathrm{Re}(\mathcal{C}_|) = y_0 - \mathrm{y}_{00} - \sigma\mathrm{Im}(e^{-i\phi}z) = y_0 - \mathrm{y}_{00} + \sigma(sy_1 - cy_2),\\ \sigma\mathrm{Im}(\mathcal{C}_|) = -h + \mathrm{Re}(e^{-i\phi}z) = cy_1 + sy_2 - h\end{aligned} \tag{6.3}$$

They are both *linear* in y-variables. Our universal element P_2 is a quadratic combination of these two generators:

$$P_2 = (y_0 - \mathrm{y}_{00})^2 + h^2 - y_1^2 - y_2^2 = -|\mathcal{C}_||^2 + 2\Big((y_0 - \mathrm{y}_{00})\mathrm{Re}(\mathcal{C}_|) - h\,\mathrm{Im}(\mathcal{C}_|)\Big) \tag{6.4}$$

For example, the boundary rings of coordinate axes are produced by the complex generators

$$\begin{array}{lll} y_2 \text{ axis}: & \mathcal{C}_|(0,\sigma|0,0) = y_0 + i\sigma z = (y_0 - \sigma y_2) + iy_1, \\ y_1 \text{ axis}: & \mathcal{C}_|(\frac{\pi}{2},\sigma|0,0) = y_0 + \sigma z = (y_0 + \sigma y_1) + i\sigma y_2 \end{array} \tag{6.5}$$

Indeed, the normal to the y_2 axis is directed along the y_1, i.e. $\phi = 0$, while normal to y_1 is directed along y_2 so that $\phi' = \frac{\pi}{2}$. Further, $\mathcal{C}_|(0,\sigma|0,0) = 0$ implies that $y_1 = 0$ and $y_0 = \sigma y_2$, while $\mathcal{C}_|(\frac{\pi}{2},\sigma'|0,0) = 0$ – that $y_2 = 0$ and $y_0 = -\sigma' y_1$.

[6] This is the same simple algebro-geometric statement, which is the origin of the old puzzle in the foundations of first-quantized string theory, see [39].

[7] It deserves emphasizing once again, that we are interested in not-generic "singular" situation, what is best illustrated by s.4 above, and all the associated peculiarities are essential.

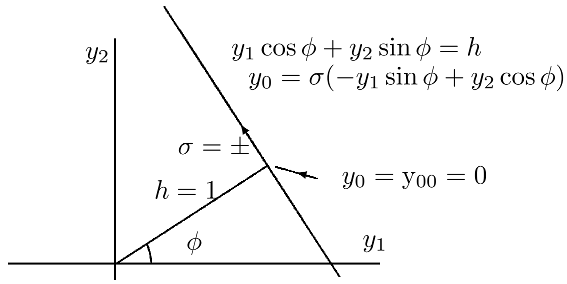

Figure 14: A single segment, a part of a straight line. Shown is its projection $\bar{\Pi} = |$ on the (y_1, y_2) plane. The line is light-like and thus is fully defined by three parameters: angle ϕ, distance h and discrete choice $\sigma = \pm$ of the y_0 direction w.r.t. direction in the (y_1, y_2) plane, denoted by arrow on the line. Such straight line in the 3d space $(y_0; y_1, y_2)$ satisfies two real-valued linear equations, which can be unified into a single complex-valued $\mathcal{C}_| = 0$, eq.(6.2). Note that ϕ is defined to be the direction of a *normal*, not of the line itself.

For generic ϕ the real and imaginary parts of $\mathcal{C}_|$ are:

$$\begin{aligned} \mathrm{Re}(\mathcal{C}_|) = 1 - cy_1 - sy_2 &\stackrel{(2.6)}{=} P_|(y_1, y_2), \\ \mathrm{Im}(\mathcal{C}_|) = \sigma y_0 + sy_1 - cy_2 &\equiv \sigma \mathcal{L}_|^\sigma \end{aligned} \tag{6.6}$$

where $c = \cos\phi$, $s = \sin\phi$ and $\mathcal{L}$ is a linear element from $\mathcal{R}_|$, satisfying the condition (2.15).

It is clear from these examples that only the real and imaginary part together, not any one of them separately, provides an adequate description of the ring. Perhaps more surprisingly, if we take any of these two elements and supplement it by P_2, we do *not* obtain a proper description of the boundary ring. Indeed, a pair $\{P_|, P_2\}$ does not contain any information about σ and can not distinguish between the two *different* boundary rings $\mathcal{R}_|^{\sigma=+}$ and $\mathcal{R}_|^{\sigma=-}$, associated with two different polygons Π which have the same projection $\bar{\Pi}$ on the (y_1, y_2) plane. As to the pair $\{\mathcal{L}_|^\sigma, P_2\}$, it specifies σ appropriately, however it does not distinguish between two different $\bar{\Pi}$(!): two parallel, but different lines with two different angle variables ϕ and $\phi + \pi$. We return to discussion of this phenomenon in s.6.3.3 below.

6.2 From a single segment to generic polygon

Given eq.(6.2), one can immediately construct a complex element of the boundary ring for any collection of intersecting straight lines:

$$\mathcal{C}_{[+\ldots+]} = \prod_{a=1}^{n} \mathcal{C}_|(\phi_a, \sigma_a | h_a, \mathrm{y}_{0a}) \tag{6.7}$$

Actually this formula is not unique, one can change some entries in the product by complex conjugates: actually there are 2^{n-1} non-equivalent possibilities,

$$\mathcal{C}_{\underbrace{[\pm\ldots\pm]}_{n}}\{\phi_a, \sigma_a | h_a, \mathrm{y}_{0a}\}, \tag{6.8}$$

where $\pm$ label the choice of $\mathcal{C}_1$ or $\overline{\mathcal{C}_|}$ at the given position in the product (6.7). Any of them can be used for description of the boundary ring. In what follows we concentrate on $\mathcal{C}_{[+\ldots+]}$, which analytically depends on z, and make additional simplifying assumptions.

When all h_a are equal, $h_a = h$, (this happens whenever projected polygon $\bar{\Pi}$ possesses an inscribed circle), then also all y_{0a} are the same and can be shifted to $\mathrm{y}_{00} = 0$, so that

$$P_2 = y_0^2 + h^2 - y_1^2 - y_2^2 \tag{6.9}$$

is always an element of the boundary ring and polynomials $\mathcal{C}$ can be divided by P_2, like it was done in s.3.3 of [1], so that the residue can be required to satisfy some constraint of our choice. For example, it can always be made linear in y_0 and satisfy the linearity condition (2.15). As an example of a different choice, z-analyticity

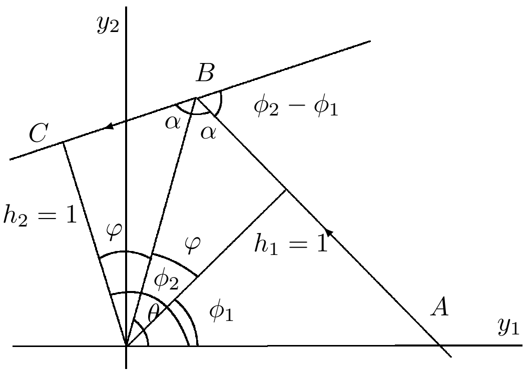

Figure 15: A pair of segments AB and BC, which form an angle ABC of the size $2\alpha = \pi + \phi_1 - \phi_2$. Both sides of the angle are at the same distance $h_1 = h_2 = 1$ from the origin. Shown also are the angle $\theta = \frac{\phi_1+\phi_2}{2}$, which defines the direction to the vertex B of the angle and $\varphi = \theta - \phi_1 = \phi_2 - \theta = \frac{\pi}{2} - \alpha = \frac{\phi_2-\phi_1}{2}$.

implies that P_2 is not involved. Most important, sometime the division procedure can be used to decrease the degree of the bound ring element: $\mathcal{C}$ defined in (6.7) has degree n in y-variables.

Since all h_a are equal, we rescale y-variables to put $h = 1$. Thus in what follows in this section $h = 1$, $\mathrm{y}_{0a} = 0$ and subscript in (6.8) is always $[+\ldots+]$. Therefore all these labels will be omitted. Instead, to further simplify the formulas, σ will be often attached as superscript to the corresponding ϕ-variable. Finally, in most cases (but not everywhere) we assume that y_0 switches direction at the vertex, i.e. $\sigma_{a+1} = -\sigma_a$ and $\sigma_a = (-)^{a-1}$ – this, however, will always be mentioned *explicitly*.

6.3 A chain of two null segments: an angle (cusp or cross) and two parallel lines

Consider first the two neighboring segments, with different angles $\phi = \phi_1$ and $\phi' = \phi_2$, which meet at a vertex and form an angle $2\alpha = \pi - (\phi' - \phi)$ (often called "cusp" in the literature on string/gauge duality; since polynomials from the boundary ring will vanish on entire two straight lines it could even better be named "cross" in this context).

6.3.1 The case of $\sigma_2 = -\sigma_1$

With all above-mentioned restrictions we have

$$\mathcal{C}_\angle = \mathcal{C}\{\phi_2^-, \phi_1^+\} = \mathcal{C}_|(\phi_2^-)\mathcal{C}_|(\phi_1^+) = \Big(y_0 + \imath(1 - e^{-\imath\phi_1}z)\Big)\Big(y_0 - \imath(1 - e^{-\imath\phi_2}z)\Big) =$$
$$= 1 + y_0^2 - 2ze^{-\imath\theta}(y_0\sin\varphi + \cos\varphi) + \big(ze^{-\imath\theta}\big)^2 = P_2 + z\bar{z} - 2ze^{-\imath\theta}(y_0\sin\varphi + \cos\varphi) + \big(ze^{-\imath\theta}\big)^2 \tag{6.10}$$

where $\phi_1 = \theta - \varphi$ and $\phi_2 = \theta + \varphi$.

For example, at $\theta = \frac{\pi}{4}$ imaginary and real part of $\mathcal{C}_\angle$ are

$$\mathrm{Im}\Big(\mathcal{C}_\angle(\theta = \frac{\pi}{4})\Big) = (y_1 - y_2)\Big(\sqrt{2}(y_0\sin\varphi + \cos\varphi) - (y_1 + y_2)\Big) \tag{6.11}$$

and

$$\mathrm{Re}\Big(\mathcal{C}_\angle(\theta = \frac{\pi}{4})\Big) = 1 + y_0^2 - \sqrt{2}(y_0\sin\varphi + \cos\varphi)(y_1 + y_2) \tag{6.12}$$

respectively. These two elements of $\mathcal{R}_\angle$ are related by addition/subtraction of P_2, one of them is quadratic while another linear in y_0, however, the coefficient in front of y_0 is proportional to $y_1 - y_2$ and condition (2.15) is not satisfied. However, this $(y_1 - y_2)$ is a common factor in front of entire expression, moreover it does not belong to $\mathcal{R}_\angle$ and can be simply thrown away – thus giving rise to an y-linear element of $\mathcal{R}_\angle$.

Since $\theta = \frac{\pi}{4}$ is not a restriction (θ can be changed by overall rotation), this linear element always exists in $\mathcal{R}_\angle^{-+}$. Because it is a procedure that we repeatedly use below, we formulate it once again. Subtracting P_2, one can convert $\mathcal{C}_\angle$ into an y_0-linear element of the boundary ring:

$$\mathcal{C}\{\phi_2^-, \phi_1^+\} - P_2 = z\bar{z} - 2ze^{-\imath\theta}(y_0\sin\varphi + \cos\varphi) + \big(ze^{-\imath\theta}\big)^2 \tag{6.13}$$

The crucial phenomenon is that the coefficient of the y_0-linear term is actually a common factor z in the whole expression. Furthermore, it is not identically zero in the ring and thus can be eliminated. This provides a *new* element of the boundary ring which in this case is automatically linear in *all* the y-variables:

$$\frac{\mathcal{C}\{\phi_2^-,\phi_1^+\}-P_2}{ze^{-\imath\theta}} = \ \bar{z}e^{\imath\theta}+ze^{-\imath\theta}-2(y_0\sin\varphi+\cos\varphi)=-2\mathcal{L}_\angle \tag{6.14}$$

Indeed, substituting $ze^{-\imath\phi}=1+i\sigma y_0$ we get:

$$\frac{1}{2}\mathcal{L}_\angle\Big\{(\theta+\varphi)^-,(\theta-\varphi)^+\Big\}\Big|_{z=(1+i\sigma y_0)e^{\imath\phi}} = \Big(\cos(\theta-\phi)-\cos\varphi\Big)+y_0\Big(\sigma\sin(\theta-\phi)-\sin\varphi\Big) \tag{6.15}$$

and this expression obviously vanishes for $\theta-\phi=\pm\varphi$ and $\sigma=\pm1$.

Note that despite we obtained it from the complex-valued character $\mathcal{C}_\angle$, this new element (6.14) is *real*:

$$\mathcal{L}_\angle^{-+}=y_0\sin\varphi+\cos\varphi-y_1\cos\theta-y_2\sin\theta=y_0\cos\alpha+\sin\alpha-y_1\cos\theta-y_2\sin\theta \tag{6.16}$$

We do not divide the r.h.s. by $\cos\alpha$ to simplify the formulas, however, this hides the singularity of the limit $\alpha\to\frac{\pi}{2}$. At other values of α the boundary ring $\mathcal{R}_\angle^{-+}$ is nicely described by the pair $(\mathcal{L}_\angle,P_2)$.

Existence of $\mathcal{L}$ is a non-trivial phenomenon. We do not need to go far away to find a situation when it does not exist: it is enough to switch from alternating σ to a constant one.

6.3.2 The case of $\sigma_2=\sigma_1$

In this case we obtain:

$$\mathcal{C}\{\phi_2^+,\phi_1^+\}=\mathcal{C}_|(\phi_2^+)\mathcal{C}_|(\phi_1^+)=\Big(y_0-\imath(1-e^{-\imath\phi_1}z)\Big)\Big(y_0-\imath(1-e^{-\imath\phi_2}z)\Big)=$$
$$=1+y_0^2-2(1-ze^{-\imath\theta}\cos\varphi)(1+\imath y_0)-\big(ze^{-\imath\theta}\big)^2=P_2+z\bar{z}-2(1-ze^{-\imath\theta}\cos\varphi)(1+\imath y_0)-\big(ze^{-\imath\theta}\big)^2 \tag{6.17}$$

We can again subtract P_2 in order to obtain an y_0-linear element of $\mathcal{R}_\angle^{++}$:

$$\mathcal{C}\{\phi_2^+,\phi_1^+\}-P_2=z\bar{z}-2(1-ze^{-\imath\theta}\cos\varphi)(1+\imath y_0)-\big(ze^{-\imath\theta}\big)^2 \tag{6.18}$$

However, the coefficient of y_0 term is now not a common factor of entire expression and can not be eliminated. A linear element exists in $\mathcal{R}_\angle^{+-}$ but not in $\mathcal{R}_\angle^{++}$.

This last part of this conclusion has a remarkable exception: $\cos\varphi=0$, i.e. $\varphi=\frac{\pi}{2}$.

6.3.3 Two parallel lines. The case of $\sigma_2=-\sigma_1$

In many non-generic examples, like Z_n-symmetric configurations of [1] or $z_2\times Z_2$-symmetric rhombus of [4] the possible building block is a pair of parallel lines, which is a particular choice of our angle with $2\alpha=0$. Moreover, both cases $\sigma_2=\pm\sigma_1$ are needed for this kind of application, even if we are interested in n-angle polygons with even n and alternated $\sigma_a=(-)^{a-1}$: for $n=4k-2$ the parallel sides will have opposite σ's, while for $n=4k$ their σ's will be the same.

Substituting $\varphi=\frac{\pi}{2}$, i.e. $\theta=\frac{\pi}{2}+\phi$ into (6.14), we obtain:

$$\mathcal{L}_{||}^{-+}=y_0-\mathrm{Re}(ze^{-i\theta})=y_0-\mathrm{Im}(ze^{-i\phi})=y_0-y_1\cos\theta-y_2\sin\theta=y_0+y_1\sin\phi-y_2\cos\phi \tag{6.19}$$

and

$$\mathcal{L}_{||}^{+-}=y_0+\mathrm{Re}(ze^{-i\theta})=y_0+\mathrm{Im}(ze^{-i\phi})=y_0+y_1\cos\theta+y_2\sin\theta=y_0-y_1\sin\phi+y_2\cos\phi \tag{6.20}$$

and $\mathcal{L}_{||}^{-\sigma\sigma}=0$ implies that

$$y_0=\sigma(-sy_1+cy_2) \tag{6.21}$$

Comparing (6.6) and (6.20), we can observe that

$$\mathcal{L}_{||}^{-\sigma\sigma}=\mathcal{L}_|^\sigma \tag{6.22}$$

This is manifestation of the fact, which we already mentioned in the end of s.6.1. Now we can formulate it in a better way: it turns out that $\mathcal{L}_|$ is not just an element of the boundary ring $\mathcal{R}_|$, it actually lies in its sub-ring:

$$\mathcal{L}_| \in \mathcal{R}_{||} \subset \mathcal{R}_| \tag{6.23}$$

Whenever the boundary $\Pi = \Pi_1 \cup \Pi_2$ is decomposed into two components, we have

$$\mathcal{R}_{\Pi_1\cup\Pi_2} \subset \mathcal{R}_{\Pi_1}, \quad \mathcal{R}_{\Pi_1\cup\Pi_2} \subset \mathcal{R}_{\Pi_2} \tag{6.24}$$

and all the elements of a polygon boundary ring naturally belong to the bigger boundary rings of its particular segments, angles, triangles etc. What we encountered, however, is a kind of an opposite phenomenon: in our attempt to build up representation of a given boundary ring, namely $\mathcal{R}_|$ we actually obtained elements of its sub-ring $\mathcal{R}_{||}$ instead of elements in generic position! We shall encounter more examples of this kind below, and one should always be careful to check what the actual nature of emerging elements is.

6.3.4 Two parallel lines. The case of $\sigma_2 = \sigma_1$

As mentioned at the very end of s.6.3.2, two parallel lines provide a practically important exception from the rule that there are no y_0-linear elements in $\mathcal{R}^{++}_\angle$. This exception, however, has a number of non-trivial properties. At $\varphi = \frac{\pi}{2}$ and $\theta = \phi + \frac{\pi}{2}$ eq.(6.18) gives:

$$\mathcal{C}^{++}_{||} - P_2 = z\bar z - 2(1 + \imath y_0) + z^2 e^{-2\imath\phi} \tag{6.25}$$

The real and imaginary parts of this complex expression are:

$$\begin{aligned}\mathrm{Re}\Big(\mathcal{C}^{++}_{||}\Big) - P_2 = z\bar z - 2 + (y_1^2 - y_2^2)\cos(2\phi) + 2y_1y_2\sin(2\phi) = \\ = -2\Big(1 - (y_1\cos\phi + y_2\sin\phi)^2\Big) \overset{(2.6)}{=} -2P_{||}(y_1, y_2)\end{aligned} \tag{6.26}$$

and

$$\mathrm{Im}\Big(\mathcal{C}^{++}_{||}\Big) = -2y_0 - (y_1^2 - y_2^2)\sin(2\phi) + 2y_1y_2\cos(2\phi) \equiv -2\mathcal{L}^{++}_{||} \tag{6.27}$$

For $\sigma_2 = \sigma_1 = -1$ the answer will differ by sign in front of y_0, and we obtain the linear element in $\mathcal{R}^{\sigma\sigma}_{||}$ in the form:

$$\mathcal{L}^{\sigma\sigma}_{||}(\phi) = y_0 - \sigma\Big(y_1y_2\cos(2\phi) + \frac{1}{2}(y_2^2 - y_1^2)\sin(2\phi)\Big) = y_0 - \sigma y_1^\phi y_2^\phi, \tag{6.28}$$

where

$$\begin{aligned} y_1^\phi &= y_1\cos\phi - y_2\sin\phi, \\ y_2^\phi &= y_1\sin\phi + y_2\cos\phi \end{aligned} \tag{6.29}$$

are rotated coordinates y_1 and y_2. In particular, for two vertical lines ($\phi = 0$) we obtain:

$$\mathcal{L}^{++}_{||} = y_0 - y_1y_2 \tag{6.30}$$

It is now obvious that what we obtained is not just an element from $\mathcal{R}_{||}$ – it actually belongs to its sub-ring $\mathcal{R}_\square$: vanishes on *four* sides of the unit square, not only on the two vertical lines, which formed our Π:

$$\mathcal{L}^{++}_{||} \in \mathcal{R}_\square \subset \mathcal{R}^{++}_{||} \tag{6.31}$$

Worse than that, in this case one can not find any element of the boundary ring $\mathcal{R}^{++}_{||}$ which could be used as a complement of P_2 in adequate description of the boundary ring: such description is available only without P_2, for example $\mathcal{C}^{++}_{||}$ in this case is a pair $\{1 - y_1^2,\ y_0 - y_1y_2\}$. This in turn means that our approach to NG solutions would not work in this case: and indeed two parallel lines with coincident σ's form an impossible Π, such *diangle* formed by two null lines is simply non-existing (while a similar diangle with $\sigma_2 = -\sigma_1$ does exist, and is an $n = 2$ version of the Z_n-symmetric configurations of [1] with (6.20) providing (together with the usual $r^2 = P_2$) an exact solution to NG equations.

6.4 Pairs of parallel lines: from square to hexagons

The boundary rings for a square and, more generally, for a rhombus can be constructed from already available building blocks in two ways: by combining two pairs of parallel lines and by combining two non-adjacent angles. Only the second one of these options is available for kite and for generic skew quadrilateral, but it is a little more complicated and we begin from analysis of the first one.

6.4.1 Square

We know already that $\mathcal{L}_{||}^{++}$ occasionally belongs to $\mathcal{R}_\square$ and we do not need to do any more calculations. However, we know this because the situation is very simple and all answers are immediately clear "from the first look". But what we need, is a kind of a systematic approach to construction of boundary rings, not relying upon accidental observations. Therefore we proceed regularly in this trivial example and use it to illustrate the general procedure. This procedure implies that we take y_0-linear elements, associated with our building blocks, multiply them and try to make them y_0-linear again by subtracting the always-available polynomials P_2 and P, $\tilde{P}$, $\tilde{\tilde{P}}$ from (2.6). If we are building the square from two pairs of parallel lines, this means that write:

$$\mathcal{L}_{||}^{--}\left(\frac{\pi}{2}\right)\mathcal{L}_{||}^{++}(0) \overset{(6.28)}{=} \left(y_0+(-y_1y_2)\right)(y_0-y_1y_2) = (y_0-y_1y_2)^2 y_0^2 - 2y_0y_1y_2 + y_1^2y_2^2 \tag{6.32}$$

Next we subtract P_2 to eliminate the term y_0^2:

$$\mathcal{L}_{||}^{--}\left(\frac{\pi}{2}\right)\mathcal{L}_{||}^{++}(0) - P_2 = -2y_0y_1y_2 + y_1^2y_2^2 - 1 + y_1^2 + y_2^2 \tag{6.33}$$

This element does not deserve the name of $\mathcal{L}_\square$, because the coefficient in front of y_0 is not constant. This coefficient does not belong to $\mathcal{R}_\square$ thus in principle we could eliminate it. Unfortunately, it is not a common factor in front of entire expression, so we can not simply get rid of it. What we can do, however, is to make use of

$$P_\square \overset{(2.6)}{=} (1-y_1^2)(1-y_2^2) \tag{6.34}$$

which is an "obvious" element of $\mathcal{R}_\square$. Adding it to (6.33) we obtain:

$$\mathcal{L}_{||}^{--}\left(\frac{\pi}{2}\right)\mathcal{L}_{||}^{++}(0) - P_2 + P_\square = -2y_1y_2y_0 + 2y_1^2y_2^2 = -2y_1y_2(y_0-y_1y_2) = -2y_1y_2\mathcal{L}_\square \tag{6.35}$$

Now the coefficient of y_0 is a common factor and can be thrown away to give

$$\mathcal{L}_\square = y_0 - y_1y_2 \tag{6.36}$$

6.4.2 Rhombus

Above procedure is immediately generalized to the case of rhombus:

$$\mathcal{L}_{||}^{--}\left(\frac{\pi}{4}+\varphi\right)\mathcal{L}_{||}^{++}\left(\frac{\pi}{4}-\varphi\right) \overset{(6.28)}{=}$$

$$= \left(y_0 + y_1y_2\cos\left(\frac{\pi}{2}+2\varphi\right) + \frac{1}{2}(y_2^2-y_1^2)\sin\left(\frac{\pi}{2}+2\varphi\right)\right)\left(y_0 - y_1y_2\cos\left(\frac{\pi}{2}-2\varphi\right) - \frac{1}{2}(y_2^2-y_1^2)\sin\left(\frac{\pi}{2}-2\varphi\right)\right) =$$

$$= \left(y_0 - y_1y_2\sin(2\varphi) + \frac{1}{2}(y_2^2-y_1^2)\cos(2\varphi)\right)\left(y_0 - y_1y_2\sin(2\varphi) - \frac{1}{2}(y_2^2-y_1^2)\cos(2\varphi)\right) =$$

$$= \left(y_0 - y_1y_2\sin(2\varphi)\right)^2 - \frac{1}{4}(y_2^2-y_1^2)^2\cos^2(2\varphi) \tag{6.37}$$

In the case o square $\varphi = \frac{\pi}{4}$ and $2\varphi = \frac{\pi}{2}$. Subtraction of P_2 converts this expression into

$$\mathcal{L}_{||}^{--}\left(\frac{\pi}{4}+\varphi\right)\mathcal{L}_{||}^{++}\left(\frac{\pi}{4}-\varphi\right) - P_2 = -2y_0y_1y_2\sin(2\varphi) + y_1^2y_2^2\sin^2(2\varphi) + y_1^2 + y_2^2 - 1 - \frac{1}{4}(y_2^2-y_1^2)^2\cos^2(2\varphi)$$

Now we need to get rid of the terms that are not divisible by y_1y_2, and again we have $P_\diamond$ to try to achieve this. Substituting $\phi_1 = \frac{\pi}{4}-\varphi$, $\phi_2 = \frac{\pi}{4}+\varphi$, $\phi_3 = \phi_1+\pi$ and $\phi_4 = \phi_2+\pi$ into the first line of (2.6), we obtain:

$$P_\diamond = \left(1-(y_1\cos\phi_1 + y_2\sin\phi_1)^2\right)\left(1-(y_1\cos\phi_2 + y_2\sin\phi_2)^2\right) =$$

$$= \left(1-\frac{1}{2}\Big(y_1(c+s)+y_2(c-s)\Big)^2\right)\left(1-\frac{1}{2}\Big(y_1(c-s)+y_2(c+s)\Big)^2\right) =$$

$$= \left(1-\frac{1}{2}\Big(y_1^2+y_2^2+2y_1y_2\cos(2\varphi)+(y_1^2-y_2^2)\sin(2\varphi)\Big)\right)\left(1-\frac{1}{2}\Big(y_1^2+y_2^2+2y_1y_2\cos(2\varphi)-(y_1^2-y_2^2)\sin(2\varphi)\Big)\right) =$$

$$= 1-y_1^2-y_2^2-2y_1y_2\cos(2\varphi)+\frac{1}{4}\left(\Big(y_1^2+y_2^2+2y_1y_2\cos(2\varphi)\Big)^2-(y_1^2-y_2^2)^2\sin^2(2\varphi)\right) \tag{6.38}$$

At intermediate stage we denoted $c=\cos\varphi$ and $s=\sin\varphi$. Now we are ready to combine:

$$\mathcal{L}_{||}^{--}\left(\frac{\pi}{4}+\varphi\right)\mathcal{L}_{||}^{++}\left(\frac{\pi}{4}-\varphi\right)-P_2+P_\diamond =$$

$$= -2y_0y_1y_2\sin(2\varphi)-2y_1y_2\cos(2\varphi)+y_1^2y_2^2\sin^2(2\varphi)+\frac{1}{4}\left(\Big(y_1^2+y_2^2+2y_1y_2\cos(2\varphi)\Big)^2-(y_1^2-y_2^2)^2\right) =$$

$$= -2y_1y_2\left(y_0\sin(2\varphi)+\cos(2\varphi)-\frac{1}{2}y_1y_2\sin^2(2\varphi)-\frac{1}{2}\Big(y_1+y_2\cos(2\varphi)\Big)\Big(y_2+y_1\cos(2\varphi)\Big)\right) =$$

$$= -2y_1y_2\left(y_0\sin(2\varphi)+\cos(2\varphi)\left(1-\frac{1}{2}(y_1^2+y_2^2)\right)-y_1y_2\right) = -2y_1y_2\sin(2\varphi)\mathcal{L}_\diamond \tag{6.39}$$

All terms, which were not divisible by y_1y_2, canceled and we finally obtain:

$$\mathcal{L}_\diamond = y_0-\frac{1}{\sin(2\varphi)}y_1y_2+\frac{\cos(2\varphi)}{\sin(2\varphi)}\left(1-\frac{1}{2}(y_1^2+y_2^2)\right) = y_0-y_1y_2\cosh\xi+\left(1-\frac{1}{2}(y_1^2+y_2^2)\right)\sinh\xi \tag{6.40}$$

where a new parameter ξ introduced, related to φ by

$$\cosh\xi = \frac{1}{\sin(2\varphi)} = \frac{1}{\cos(2\phi_1)}, \quad \sinh\xi = \frac{\cos(2\varphi)}{\sin(2\varphi)} = \tan(2\phi_1) \tag{6.41}$$

Thus we *derived* an expression for $\mathcal{L}_\diamond$. It is *canonical* in the sense that this is the only element of $\mathcal{R}_\diamond$, which is linear in y_0 and satisfies (2.15). Moreover, it has degree $2=\frac{n}{2}$ in y-variables! Any other element of degree 2 in $\mathcal{R}_\diamond$ can be obtained by adding P_2 with some constant coefficient. It is within this 1-parametric family

$$\mathcal{L}_\diamond+\mu P_2 = 0 \tag{6.42}$$

that we expect to find the solution to Plateau problem (since we know from section 5 that for $n=4$ the solution is quadratic in y):

$$\mathcal{S}_\diamond \sim \mathcal{L}_\diamond+\mu_\diamond P_2 \tag{6.43}$$

the value $\mu_\diamond$ can not be found by the study of the boundary ring alone: it is defined either by NG equations or by minimization of regularized action w.r.t. to μ-variable. Since we actually know what $\mathcal{S}_\diamond$ is, we can use this answer, eq.(5.36),

$$\mathcal{S}_\diamond \stackrel{(5.36)}{\sim} y_0-y_1y_2\cosh\xi+\frac{1}{2}(1-y_0^2)\sinh\xi, \tag{6.44}$$

to get:

$$\mu_\diamond = -\frac{1}{2}\sinh\xi \tag{6.45}$$

6.4.3 A two-parametric family of hexagons

If instead of two pairs of parallel lines we consider three, what we obtain will be a hexagon. It will be not a generic hexagon with inscribed circle,[8] which form a family with $n-1=5$ parameters (), but a 2-parametric sub-family, which, however, contains the Z_6-symmetric hexagon, considered in [1].

[8] If conditions (1.2) of AdS_3-embedding are not imposed, hexagons form a $3n-8=10$-parametric family: $3n$ coordinates (y_1,y_2,y_3) of $n=6$ vertices minus 3 parallel transports, minus 3 rotations, minus one rescaling and minus one constraint $\sum_a \sigma_a l_a = 0$ which guarantees that Π formed from null-segments closes in y_0 direction. If only space-flatness condition $y_3=0$ is imposed, the space of relevant hexagons reduces to $2n-5=7$ dimensions. Inscribed-circle condition (it makes sense only if $y_3=0$) imposes $n-4$ extra constraints and brings the dimension down to $n-1=5$: n angles ϕ_a minus one common rotation.

We assume that the first (and thus also the forth) side of the hexagon is parallel to the y_2-axis, $\phi_1 = 0$, $\phi_4 = \pi$ – this fixes rotation freedom. Remaining two parameters are $\phi_2 = \phi$ and $\phi_3 = \pi - \phi'$. We denote their sines and cosines by $c = \cos\phi = \cos\phi_2$, $s = \sin\phi = \sin\phi_2$, $c' = \cos\phi' = -\cos\phi_3$, $s' = \sin\phi' = \sin\phi_3$. This time we should use $\mathcal{L}_{||}^{-+}$ and $\mathcal{L}_{||}^{+-}$ rather than $\mathcal{L}_{||}^{++}$ as the building blocks.

$$\mathcal{L}_{||}^{-+}(\phi_3)\mathcal{L}_{||}^{+-}(\phi_2)\mathcal{L}_{||}^{-+}(\phi_1) \overset{(6.19)\&(6.20)}{=} (y_0 + s'y_1 + c'y_2)(-y_0 + sy_1 - cy_2)(y_0 - y_2) =$$
$$-y_0^3 + y_0^2\Big((s-s')y_1 + (1-c-c')y_2\Big) +$$
$$+\, y_0\Big(ss'y_1^2 + \big(\sin(\phi-\phi') + (s'-s)\big)y_1y_2 + (c+c'-cc')y_2^2\Big) + \Big(-ss'y_1^2y_2 - \sin(\phi-\phi')y_1y_2^2 + cc'y_2^3\Big) \tag{6.46}$$

This time we get an element of $\mathcal{R}_{hexa}$, which is cubic in y-variables, in particular it is cubic in y_0 In order to obtain an y_0-linear expression we need to subtract P_2, multiplied by a polynomial which is not just a constant, but contains also a first power of y_0. Note, however, that since we are multiplying $\mathcal{L}_{||}^{-+}$ instead of $\mathcal{L}_{||}^{++}$, the product has power $n/2 = 3$ in *all* of the y-variables, and thus we can not make use of polynomials (2.6) in order to further simplify it: all these polynomials are of degree $n = 6 > 3$.

$$\mathcal{L}_{hexa} = \mathcal{L}_{||}^{-+}(\phi_3)\mathcal{L}_{||}^{+-}(\phi_2)\mathcal{L}_{||}^{-+}(\phi_1) + \Big(y_0 + (s-s')y_1 + (c+c'-1)y_2\Big)P_2 =$$
$$= y_0\Big(1 - (1-ss')y_1^2 + \big(\sin(\phi-\phi') - (s-s')\big)y_1y_2 - (1-c)(1-c')y_2^2\Big) + \tag{6.47}$$
$$+\Big((s-s')y_1^3 + (1-c-c'-ss')y_1^2y_2 + \big(s-s'-\sin(\phi-\phi')\big)y_1y_2^2 + (1-c)(1-c')y_2^2 + (s'-s)y_1 + (c+c'-1)y_2\Big)$$

Note that this time $\mathcal{L}_{hexa}$ is linear only in y_0, it satisfies (2.15), but the coefficient in front of y_0 is non-trivial function of y_1 and y_2 which can *not* be eliminated.

This expression is considerably simplified if we restrict to a $Z_2 \times Z_2$-symmetric one-parametric family of hexagons with $\phi' = \phi$. Then

$$\mathcal{L}_{hexa} = y_0\Big(1 - c^2y_1^2 - (1-c)^2y_2^2\Big) + y_2\Big(-c(2-c)y_1^2 + (1-c)^2y_2^2 + (2c-1)\Big) \tag{6.48}$$

In the case of Z_6-symmetry, when $\phi' = \phi = \frac{\pi}{3}$ and $c = \frac{1}{2}$, it further simplifies to

$$\mathcal{L}_{hexa} = y_0\left(1 - \frac{1}{4}(y_1^2 + y_2^2)\right) - \frac{1}{4}y_2(3y_1^2 - y_2^2) \tag{6.49}$$

This expression is familiar from [1], and now we derived it applying a systematical, constructive and generalizable method.

For hexagons the full family of y-cubic ($n/2 = 3$) elements in $\mathcal{R}_{hexa}$ is 4-parametric:

$$\Big\{\mathcal{L}_{hexa} + (\mu^\lambda y_\lambda)P_2 + \nu P_2\Big\} \tag{6.50}$$

We know from [1] that exact solution to Plateau problem does *not* lie entirely in this space, but

$$\mu_{hexa}^\lambda = 0, \quad \nu_{hexa} = 0, \quad \mathcal{S}_{hexa} \approx \mathcal{L}_{hexa} + (\mu^\lambda y_\lambda)P_2 + \nu P_2 \tag{6.51}$$

provides a nice first approximation, which can be further improved by methods of s.3 – with μ promoted to a power series.

6.5 Combining angles

Instead of combining parallel lines, we can combine angles. This enlarges the set of possible configurations and is simply a necessary thing to do for description of generic asymmetric configurations, like 3-parametric family of skew quadrilaterals and its 2-parametric sub-family of *kites* at $n = 4$. Rhombi and square are further restrictions of this family to 1- and 0-parametric sub-sets. Consideration of multiple angles is straightforward, however a new phenomenon arises: particular element of the boundary ring which we obtain can depend on the choice of angles in the polygon, but canonical elements like $\mathcal{L}$ will, of course, coincide. A variety of angle variables appearing in calculations is shown in combined Fig.16.

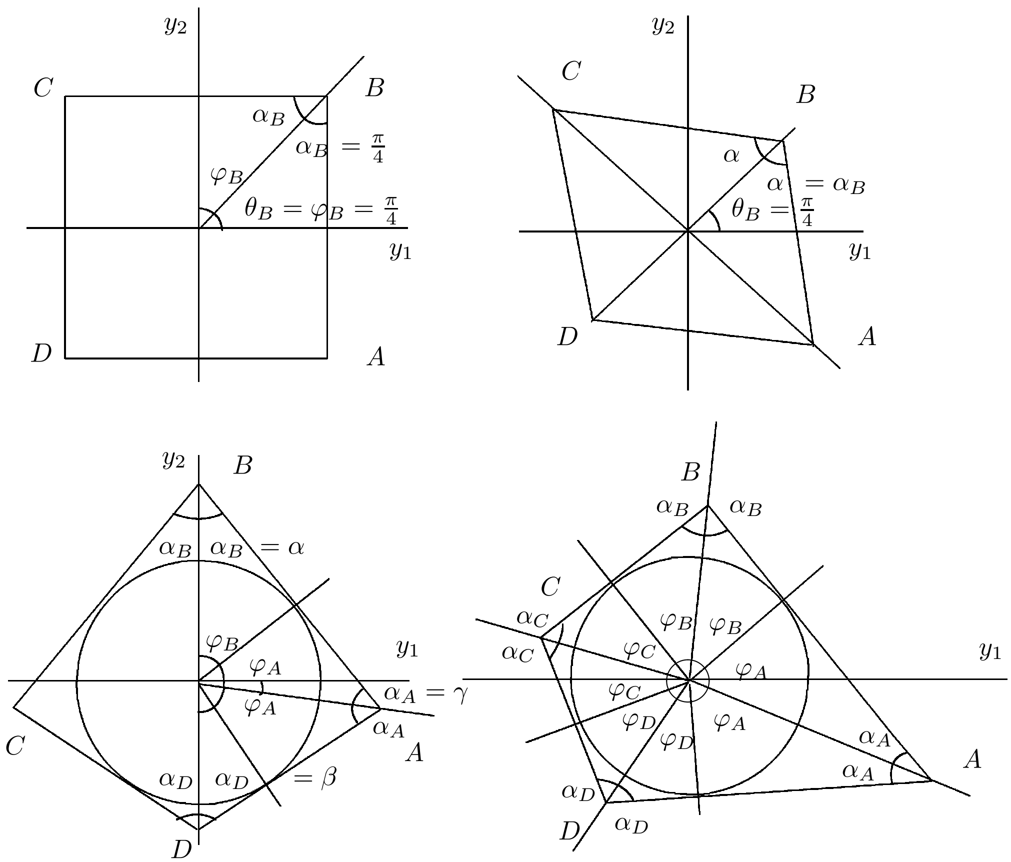

Figure 16: The four embedded families: square, rhombus, kite and generic skew quadrilateral with inscribed circle, considered in s.6.5. Shown are various angle variables used in the text. Vertices are labeled counterclockwise by alphabetically ordered capital letters, directions to corresponding vertices are denoted through θ, directions of normals – by ϕ (not shown in *this* picture),– angles between these normals and vertex directions – by φ,– finally, the angles of polygons are 2α. Obvious relations are: $\alpha_a + \varphi_a = \frac{\pi}{2}$, $\theta_{a+1} - \theta_a = \varphi_{a+1} + \varphi_a$ Relations involving ϕ's depends on the labeling of polygon sides. If vector (external momentum) $\mathbf{p}_a$ points from vertex a to vertex $a+1$, i.e. the vertex a is at the intersection of sides a and $a-1$, then $\theta_a - \varphi_a = \phi_{a-1}$ and $\theta_a + \varphi_a = \phi_a$.

6.5.1 Square

As usual, we begin from the simplest case: the square. This time we want to obtain $\mathcal{L}_\square \in \mathcal{R}_\square$ from two boundary rings $\mathcal{R}_\angle^{-+}$, associated with two *opposite* right angles, say, at vertices B and D. Following our standard procedure, we multiply the canonical $\mathcal{L}$ elements of these two rings, then subtract P_2 in order to eliminate the y_0^2-term and afterwards look at the coefficient in front of y_0: if it is not constant we add more "obvious" elements (2.6) to make this coefficient into a common factor and then throw it away. Actually, the last step will appear unnecessary in the study of a pair of angles (this is *a priori* obvious because the degree of appearing polynomials will be lower than n, and polynomials (2.6) can not mix with them).

Throughout this subsection we use the following notation:

$$\mathcal{L}_\angle^\sigma(\theta|\alpha) \stackrel{(6.16)}{=} \sigma y_0\cos\alpha + \sin\alpha - y_1\cos\theta - y_2\sin\theta \tag{6.52}$$

We remind that θ denotes direction to the vertex of the angle, while its size is 2α.

In the case of square $2\alpha = \frac{\pi}{2}$ and we locate two opposite angles at $\theta_B = \frac{\pi}{4}$ and $\theta_D = \frac{5\pi}{4}$. Then

$$\begin{aligned}\mathcal{L}_\angle^+\left(\frac{\pi}{4}\Big|\frac{\pi}{4}\right)\mathcal{L}_\angle^+\left(\frac{5\pi}{4}\Big|\frac{\pi}{4}\right) &= \frac{1}{2}\Big(y_0+1-y_1-y_2\Big)\Big(y_0+1+y_1+y_2\Big) = \\ = \frac{1}{2}\Big((y_0+1)^2-(y_1+y_2)^2\Big) &= \frac{1}{2}\Big(P_2+2(y_0-y_1y_2)\Big) = \frac{1}{2}P_2+\mathcal{L}_\square \end{aligned}\tag{6.53}$$

$\mathcal{L}_\square = y_0 - y_1y_2$ is our familiar expression, both the $\mathcal{L}$-element of $\mathcal{R}_\square$ and exact solution $S_\square$ to the AdS Plateau problem.

6.5.2 Rhombus

In the case of rhombus we keep θ's the same, $\theta_B = \frac{\pi}{4}$ and $\theta_D = \frac{5\pi}{4}$, but angles at the vertices are now not restricted to be $\frac{\pi}{4}$. Then

$$\begin{aligned}\mathcal{L}_\angle^+\left(\frac{\pi}{4}\Big|\alpha\right)\mathcal{L}_\angle^+\left(\frac{5\pi}{4}\Big|\alpha\right) &= \left(y_0\cos\alpha+\sin\alpha-\frac{1}{\sqrt{2}}(y_1+y_2)\right)\left(y_0\cos\alpha+\sin\alpha+\frac{1}{\sqrt{2}}(y_1+y_2)\right) = \\ &= (y_0\cos\alpha+\sin\alpha)^2-\frac{1}{2}(y_1+y_2)^2 = \frac{1}{2}P_2+\left(\frac{1}{2}(y_0^2-1)\cos(2\alpha)+y_0\sin(2\alpha)-y_1y_2\right) = \\ &= P_2\cos^2\alpha+\left\{y_0\sin(2\alpha)-y_1y_2-\left(1-\frac{1}{2}(y_1^2+y_2^2)\right)\cos(2\alpha)\right\} = P_2\cos^2\alpha+\mathcal{L}_\diamond\sin(2\alpha)\end{aligned}\tag{6.54}$$

For comparison with the other formulas for $\mathcal{L}_\diamond$, like (6.40), one should keep in mind that $\alpha = \frac{\pi}{2}-\varphi$, so that $\sin(2\alpha) = \sin(2\varphi)$ and $\cos(2\alpha) = -\cos(2\varphi)$.

Finally, if we use in this formula another angle of the rhombus $\alpha' = \pi - \alpha$ instead of α, then $\sin(2\alpha)$ changes sign. However, simultaneously one should change σ to $-\sigma$, since the starting side of the rhombus in above derivation has also changed. Changing sign of σ is equivalent to changing sing of y_0, thus the product $y_0\sin(2\alpha) = \sigma y_0\sin(2\alpha) = (-\sigma)y_0\sin(2(\pi-\alpha))$ does not change and $\mathcal{L}_\diamond$ remains the same – as it should, since it is a canonically defined element of the boundary ring $\mathcal{R}_\diamond$.

6.5.3 Kite

In the case of kite we can consider two essentially inequivalent choices of opposite angles: $\alpha = \alpha_B$ and $\beta = \alpha_D$ or $\gamma = \alpha_A$ and $\gamma = \alpha_C = \alpha_A = \frac{\pi}{2} - \frac{\alpha+\beta}{2}$. The corresponding angles θ will also be different: either $\theta_B = \frac{\pi}{2}$ and $\theta_D = \frac{3\pi}{2}$ or $\theta_A = \frac{\alpha-\beta}{2}$ and $\theta_C = \pi - \theta_A$.

A product of two y-linear elements $\mathcal{L}_\angle$ is usually quadratic in y and we denote it $\mathcal{Q}$. Thus in the case of kite we are interested in two different quantities $\mathcal{Q} \in \mathcal{R}_{kite}$:

$$\begin{aligned}\mathcal{Q}_{BD} = \mathcal{L}_\angle^+\Big(\theta_D\Big|\alpha_D\Big)\mathcal{L}_\angle^+\Big(\theta_B\Big|\alpha_B\Big) = \mathcal{L}_\angle^+\Big(\frac{3\pi}{2}\Big|\beta\Big)\mathcal{L}_\angle^+\Big(\frac{\pi}{2}\Big|\alpha\Big) &\stackrel{(6.16)}{=} \Big(y_0\cos\beta+\sin\beta+y_2\Big)\Big(y_0\cos\alpha+\sin\alpha-y_2\Big) = \\ = y_0^2\cos\alpha\cos\beta+y_0\Big(\sin(\alpha+\beta)+y_2(\cos\alpha-\cos\beta)\Big)&+(\sin\beta+y_2)(\sin\alpha-y_2)\end{aligned}\tag{6.55}$$

and

$$\mathcal{Q}_{AC} = \mathcal{L}_\angle^-\Big(\theta_C\Big|\alpha_C\Big)\mathcal{L}_\angle^-\Big(\theta_A\Big|\alpha_A\Big) = \mathcal{L}_\angle^-\Big(\pi-\frac{\alpha-\beta}{2}\Big|\gamma\Big)\mathcal{L}_\angle^-\Big(\frac{\alpha-\beta}{2}\Big|\gamma\Big) \stackrel{(6.16)}{=}$$

$$= \Big(- y_0 \cos\gamma + \sin\gamma + y_1 \cos\frac{\alpha-\beta}{2} - y_2 \sin\frac{\alpha-\beta}{2}\Big)\Big(- y_0 \cos\gamma + \sin\gamma - y_1 \cos\frac{\alpha-\beta}{2} - y_2 \sin\frac{\alpha-\beta}{2}\Big) =$$

$$= \Big(y_0 \sin\frac{\alpha+\beta}{2} + y_2 \sin\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta}{2}\Big)^2 - y_1^2 \cos^2\frac{\alpha-\beta}{2} =$$

$$= y_0^2 \sin^2\frac{\alpha+\beta}{2} + \cos^2\frac{\alpha+\beta}{2} - y_0\sin(\alpha+\beta) + 2y_0y_2 \sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} - 2y_2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} -$$

$$- y_1^2\cos^2\frac{\alpha-\beta}{2} + y_2^2 \sin^2\frac{\alpha-\beta}{2} \;=\; P_2\cos^2\frac{\alpha-\beta}{2} + \mathcal{Q}_{BD} \qquad (6.56)$$

Thus the two ways of construction provides us with two different elements of the boundary ring. They both belong to the family $\mathcal{Q}_{BD} + \nu P_2$, consisting of all the elements of $\mathcal{R}_{kite}$ of degree 2. Expression (6.55) is already familiar to us: it appeared in (5.47) and we also know from there how exact solution to Plateau problem is embedded into this family:

$$\mathcal{S}_{kite} \overset{(5.47)}{\sim} \mathcal{Q}_{BD} - \frac{1}{2}P_2\cos(\alpha-\beta) \qquad (6.57)$$

In order to convert $\mathcal{Q}_{BD}$ into a y_0-linear expression $\mathcal{L}_{kite}$ we need to subtract $P_2\cos\alpha\cos\beta$. However in the resulting

$$\sin(\alpha+\beta)\mathcal{L}_{kite} = \mathcal{Q}_{BD} - P_2\cos\alpha\cos\beta = y_0\Big(\sin(\alpha+\beta) + y_2(\cos\beta - \cos\alpha)\Big) - \cos(\alpha+\beta) +$$
$$+ y_2(\sin\alpha - \sin\beta) + y_1^2\cos\alpha\cos\beta - y_2^2(1-\cos\alpha\cos\beta) \qquad (6.58)$$

the coefficient in front of y_0 is non-trivial function of y_2 and it *can not* be eliminated. Still such $\mathcal{L}_{kite}$ satisfies the condition (2.15). if we parameterize the family of quadratic elements in $\mathcal{R}_{kite}$ canonically: $\Big\{\mathcal{L}_{kite} + \mu P_2\Big\}$ then exact solution $\mathcal{S}_{kite}$ is associated with

$$\mu_{kite} = \frac{1}{2}\frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)} \qquad (6.59)$$

In the particular case of $\alpha = \beta$ kite becomes rhombus and we reproduce (6.45):

$$\mu_{kite}\big|_{\alpha=\beta} = \frac{1}{2}\frac{\cos(2\alpha)}{\sin(2\alpha)} = -\frac{1}{2}\frac{\cos(2\varphi)}{\sin(2\varphi)} = \mu_\diamond \qquad (6.60)$$

6.5.4 Generic skew quadrilateral

As basic variables, parameterizing the skew quadrilateral (possessing an inscribed circle) we take the four angles φ_A, φ_B, φ_C, φ_D. Actually these are three independent variables, since $\varphi_A + \varphi_B + \varphi_C + \varphi_D = \pi$. The angles $2\alpha_A$, $2\alpha_B$, $2\alpha_C$ and $2\alpha_D$ of the quadrilateral are easily expressed through these φ's:

$$\alpha_a = \frac{\pi}{2} - \varphi_a \qquad (6.61)$$

The normals directions ϕ_a and those of the vertices θ_a are also expressed through φ_a, provided one fixes the freedom of overall rotation in the (y_1, y_2) plane. In this subsection we do this by putting $\phi_1 = 0$, so that the side AB is parallel to ordinate axis, see Fig,16. Then

$$\begin{array}{c} \phi_1 = 0, \quad \phi_2 = 2\varphi_B, \quad \phi_3 = 2\varphi_B + \varphi_C, \quad \phi_4 = -2\varphi_A - \varphi_D \\ \theta_A = -\varphi_A, \quad \theta_B = \varphi_B, \quad \theta_C = 2\varphi_B + \varphi_C, \quad \theta_D = 2\varphi_D - \varphi_A \end{array} \qquad (6.62)$$

Like kite, the boundary ring for generic skew quadrilateral can be obtained from rings for two different pairs of angles: B and D or A and C.

$$Q_{BD} = \mathcal{L}^+_\angle(\theta_D|\alpha_D)\mathcal{L}^+_\angle(\theta_B|\alpha_B) \overset{(6.16)}{=}$$

$$= \Big(y_0\cos\alpha_D + \sin\alpha_D - y_1\cos\theta_D - y_2\sin\theta_D\Big)\Big(y_0\cos\alpha_B + \sin\alpha_B - y_1\cos\theta_B - y_2\sin\theta_B\Big) =$$

$$= y_0^2\cos\alpha_B\cos\alpha_D + y_1^2\cos\theta_B\cos\theta_D + y_2^2\sin\theta_B\sin\theta_D - y_1y_2\sin(\theta_B+\theta_D) + y_0\sin(\alpha_B+\alpha_D) + \sin\alpha_B\sin\alpha_D -$$

$$-y_0y_1\Big(\cos\alpha_B\cos\theta_D+\cos\alpha_D\cos\theta_B\Big)-y_0y_2\Big(\cos\alpha_B\sin\theta_D+\cos\alpha_D\sin\theta_B\Big)-$$

$$-y_1\Big(\sin\alpha_B\cos\theta_D+\sin\alpha_D\cos\theta_B\Big)-y_2\Big(\sin\alpha_B\sin\theta_D+\sin\alpha_D\sin\theta_B\Big) \tag{6.63}$$

Similarly we can define

$$Q_{AC}=\mathcal{L}^-_\angle(\theta_C|\alpha_C)\mathcal{L}^-_\angle(\theta_A|\alpha_A) \tag{6.64}$$

It is given by the same formula with (B,D) changed for (A,C) and the sign of y_0 reversed (because the starting segment is now different and therefore $\mathcal{L}^-_\angle$ is used instead of $\mathcal{L}^+_\angle$). Both these quantities can be used to find the y_0-linear element $\mathcal{L}$:

$$\begin{aligned}Q_{BD}-P_2\cos\alpha_B\cos\alpha_D&=\sin(\alpha_B+\alpha_D)\mathcal{L}_{quadri},\\ Q_{AC}-P_2\cos\alpha_A\cos\alpha_C&=\sin(\alpha_A+\alpha_C)\mathcal{L}_{quadri}\end{aligned} \tag{6.65}$$

The fact that $\mathcal{L}$ is the same in both cases is a direct, but somewhat tedious consistency check. Both expressions can be considered as explicit expression for $\mathcal{L}_{quadri}$ – but written in terms of two different sets of independent parameters: $(\alpha_B,\alpha_D,\theta_B,\theta_D)$ in one case and $(\alpha_A,\alpha_C,\theta_A,\theta_C)$ in the other.

This $\mathcal{L}$ is exactly the $\mathcal{L}_{quadri}$ which appeared in eq.(5.4.4), which describes its relation to exact solution of *AdS* Plateau problem for generic skew quadrilateral.

6.6 Summary

We now give a short summary of our consideration of the boundary rings.

6.6.1 Boundary ring and Plateau problem

Suggested strategy is to represent the ring $\mathcal{R}_\Pi$ by *canonical* element $\mathcal{L}_\Pi$, which is linear in y_0 and satisfies the condition (2.15):

$$\mathcal{L}_\Pi=y_0\Big(1+O(y_1,y_2)\Big)+\mathcal{K}_\Pi(y_1,y_2) \tag{6.66}$$

For $\bar\Pi$ possessing an inscribed circle and thus a degree-two element $P_2\in\mathcal{R}_\Pi$, such element can be constructed from the product of complex-valued generators $\mathcal{C}_| \in \mathcal{R}_|$ of individual segments and eliminating higher powers of y_0 by subtracting P_2 with various coefficients. In this way, however, we obtain a polynomial of degree n in y_1 and y_2 which is not unique, since one can always combine it with the "obvious" $\mathcal{P}_\Pi\in\mathcal{R}_\Pi$, see eq.(2.6), which also has degree n. Worse than that, this polynomial can not serve as $\mathcal{L}_\Pi$ because it does not necessarily satisfy (6.66). In the case when y_0 in Π flips (changes direction) at every vertex, one can always adjust the combination with P_Π in such a way that a common multiplier of degree $n/2$ factors out, and after throwing it away (what is possible because this expression is not identical zero in $\mathcal{R}_\Pi$) we finally obtain the $\mathcal{L}_\Pi$, which turns out to be of degree $n/2$ in y_1 and y_2. This $\mathcal{L}_\Pi$ can be also constructed straightforwardly from building blocks $\mathcal{L}^{\mp\pm}_\angle$, associated with $n/2$ non-adjacent angles of Π instead of its n sides. Since $\mathcal{L}^{\mp\pm}_\angle$ is itself linear in all y-variables, the product of such building blocks provides an element of degree $n/2$ and modulo P_2 it is linear in y_0, as requested. It turns out that it automatically (after appropriate rescaling) satisfied (6.66).

Thus canonical element $\mathcal{L}_\Pi$

• is linear in y_0,

$$\mathcal{L}_\Pi=y_0Q_\Pi(y_1,y_2)-\mathcal{K}_\Pi(y_1,y_2); \tag{6.67}$$

• satisfies (6.66), i.e.

$$Q_\Pi(y_1,y_2)=1+O(y_1,y_2); \tag{6.68}$$

• is of degree $n/2$ in y_1 and y_2, more precisely $\mathcal{K}_\Pi$ is of degree $n/2$ and Q_Π is of degree $n/2-1$.

Such element is unique, up to overall rotation of the (y_1,y_2) plane. Unfortunately, there is no distinguished way to fix this freedom and historically it was done in different ways in different particular cases. Among existing options are: $\theta_B=\frac{\pi}{4}$ (square and rhombus in [4]), $\theta_B=\frac{\pi}{2}$ (kite, a natural choice), $\phi_1=0$ (square and other Z_n-symmetric configurations of [1], generic quadrilateral and skew hexagons in this paper). Vertex B is the one where y_0 is takes its maximal positive value. Still another option is to require that the coefficient

in front of $y_1^{n/2}$ – the maximal power of y_1 – vanishes. Rotational freedom should be taken into account in comparison of different formulas in this paper.

Entire family of elements of degree $n/2$ in $\mathcal{R}_\Pi$ is spanned by polynomials of degree $n-2$ of three variables y_0, y_1, y_2:

$$\Big\{\mathcal{L}_\Pi + \mu(\mathbf{y})P_2\Big\} \tag{6.69}$$

The suggestion is to look for the first approximation to solution of the *AdS* Plateau problem within this set – finding the optimal *point* μ_Π in this *moduli space* (made of polynomials), either directly from NG equations or from minimization of the regularized action over μ *a la* [13]. Then this approximate solution can be further perturbed, as described in [1] and s.3 above.

6.6.2 List of the simplest $\mathcal{L}_\Pi$

We now list briefly the simplest examples of $\mathcal{L}$, obtained in the previous subsections what provides a general look on the problem.

Single segment:

$$\mathcal{C}_|^\pm(\phi) = 1 \pm \imath y_0 - ze^{-i\phi} \tag{6.70}$$

is the complex generator, consisting of two real ones:

$$\mathrm{Re}(\mathcal{L}_|^C) = 1 - cy_1 - sy_2 \overset{(2.6)}{=} P_|(y_1, y_2), \quad c = \cos\phi,\ s = \sin\phi \tag{6.71}$$

and

$$\mathcal{L}_|^\pm = \mathrm{Re}(\mathcal{L}_|^{C\pm}) = \pm y_0 + sy_1 - cy_2 \tag{6.72}$$

P_Π does not contain y_0 and is independent of the sign σ. $\mathcal{L}_|^\pm$ is actually an element of a special sub-ring in $\mathcal{R}_|$,

$$\mathcal{L}_|^\pm \in \mathcal{R}_{||}^{\mp\pm} \subset \mathcal{R}_|^\pm, \tag{6.73}$$

and does not adequately represent $\mathcal{R}_|$ itself. Angle ϕ specifies the direction of a *normal* to the segment, direction of the segment itself is $\phi + \frac{\pi}{2}$.

Two segments, forming an angle of the size 2α with flipping $\sigma_2 = -\sigma_1 = 1$:

$$\mathcal{L}_\angle(\theta|\alpha) = \mathcal{L}_\angle^{-+} = y_0\cos\alpha + \sin\alpha - y_1\cos\theta - y_2\sin\theta \tag{6.74}$$

θ defines the direction to the angle's vertex. It is related to the single-segment quantities by

$$ze^{-i\theta}\mathcal{L}_\angle(\theta|\alpha) = \mathcal{C}_|^+(\theta-\varphi)\mathcal{C}_|^-(\theta+\varphi) - P_2 \tag{6.75}$$

Here the normal directions are $\phi_1 = \theta - \varphi$ and $\phi_2 = \theta + \varphi$, so that $\varphi = \frac{\pi}{2} - \alpha$. In particular, for **two parallel segments** we have:

$$\mathcal{L}_{||}^{\mp\pm} = y_0 \mp (y_1\sin\phi + y_2\cos\phi) \tag{6.76}$$

Such combination appears in description of symmetric n-angle polygons with $n = 4k-2$, including $n = 2$ (see s.2.1 of [1]) and $n = 6$ (hexagon). For $n = 4k$ another combination of σ's is needed, then:

$$\mathcal{L}_{||}^{\pm\pm} = y_0 \mp \left(y_1y_2\cos(2\phi) + \frac{1}{2}(y_2^2 - y_1^2)\sin(2\phi)\right) \tag{6.77}$$

Square and rhombus belong to this class of examples.

Four segments can be described as a combination of two non-adjacent angles with alternating σ:

$$\mathcal{L}_{quadri} = \frac{1}{\sin(\alpha_1+\alpha_3)}\Big(\mathcal{L}_\angle^{-+}(\theta_3|\alpha_3)\mathcal{L}_\angle^{-+}(\theta_1|\alpha_1) - P_2\cos\alpha_1\cos\alpha_2\Big) =$$

$$= y_0 \left(1 - \frac{\cos\alpha_1 \cos\theta_3 + \cos\alpha_3 \cos\theta_1}{\sin(\alpha_1+\alpha_3)}\, y_1 - \frac{\cos\alpha_1 \sin\theta_3 + \cos\alpha_3 \sin\theta_1}{\sin(\alpha_1+\alpha_3)}\, y_2 \right) +$$
$$+ \frac{\cos\theta_1 \cos\theta_3 + \cos\alpha_1 \cos\alpha_3}{\sin(\alpha_1+\alpha_3)}\, y_1^2 + \frac{\sin\theta_1 \sin\theta_3 + \cos\alpha_1 \cos\alpha_3}{\sin(\alpha_1+\alpha_3)}\, y_2^2 - \frac{\sin(\theta_1+\theta_3)}{\sin(\alpha_1+\alpha_3)}\, y_1 y_2 -$$
$$- \frac{\sin\alpha_1 \cos\theta_3 + \sin\alpha_3 \cos\theta_1}{\sin(\alpha_1+\alpha_3)}\, y_1 - \frac{\sin\alpha_1 \sin\theta_3 + \sin\alpha_3 \sin\theta_1}{\sin(\alpha_1+\alpha_3)}\, y_2 - \frac{\cos(\alpha_1+\alpha_3)}{\sin(\alpha_1+\alpha_3)} \tag{6.78}$$

For particular sub-families this expression simplifies:
Kite, $\theta_3 - \theta_1 = \pi$: (we also put $\theta_1 = \frac{\pi}{2}$, i.e. $y_{1,2} = y_{1,2}^{\theta_1 - \pi/2}$)

$$\mathcal{L}_{kite} = y_0 \left(1 + \frac{\sin\frac{\alpha_1-\alpha_3}{2}}{\cos\frac{\alpha_1+\alpha_3}{2}}\, y_2 \right) + \frac{\sin\frac{\alpha_1-\alpha_3}{2}}{\sin\frac{\alpha_1+\alpha_3}{2}}\, y_2 + \frac{\cos\alpha_1\cos\alpha_3}{\sin(\alpha_1+\alpha_3)}\, y_1^2 - \frac{1-\cos\alpha_1\cos\alpha_3}{\sin(\alpha_1+\alpha_3)}\, y_2^2 - \frac{\cos(\alpha_1+\alpha_3)}{\sin(\alpha_1+\alpha_3)} \tag{6.79}$$

Rhombus, i.e. kite with $\alpha_3 = \alpha_1 = \alpha$:

$$\mathcal{L}_\diamond = y_0 + \frac{1+\cos(2\alpha)}{\sin(2\alpha)}(y_1^2 + y_2^2) - \frac{1}{\sin(2\alpha)} y_2^2 - \frac{\cos(2\alpha)}{2\sin(2\alpha)} \tag{6.80}$$

Rotation by $\pi/4$, $(y_1, y_2) \to \frac{1}{\sqrt{2}}(y_1 - y_2, y_1 + y_2)$, and substitution $2\alpha = \frac{\pi}{2} - 2\phi$ convert this into

$$\mathcal{L}_\diamond = y_0 - \frac{1}{\cos(2\phi)} y_1 y_2 - \frac{\sin(2\phi)}{\cos(2\phi)} \left(1 - \frac{1}{2}y^2\right) = y_0 - y_1 y_2 \cosh\xi - \left(1 - \frac{1}{2}y^2\right)\sinh\xi =$$
$$= y_0 - \frac{1+b^2}{1-b^2} y_1 y_2 - \frac{2b}{1-b^2}\left(1 - \frac{1}{2}y^2\right) \tag{6.81}$$

with $y^2 = y_1^2 + y_2^2$, $\cosh\xi = \frac{1}{\cos(2\phi)}$, $\sinh\xi = \frac{\sin(2\phi)}{\cos(2\phi)}$ and $b = \tan\phi$.
Square, i.e. rhombus with $2\alpha = \frac{\pi}{2}$:

$$\mathcal{L}_\square = y_0 - y_1 y_2 \tag{6.82}$$

These examples are concisely represented in the following table. Its first part contains examples which are symmetric under $z \to -z$ accompanied by either $y_0 \to y_0$ or $y_0 \to -y_0$. Examples in the second part of the table do not have this symmetry. An element $\mathcal{A} - \mathcal{B}$ of $\mathcal{R}_\Pi$ is often written as $\mathcal{A} = \mathcal{B}$.

$\Pi =$ set of n segments	set of σ's	$\mathcal{L}_\Pi \in \mathcal{R}_\Pi$ (linear in y_0, of degree $\frac{n}{2}$ in y_1, y_2)	$\mathcal{S}_\Pi \in \overline{\mathcal{R}_\Pi}$, $\mathcal{S}_\Pi = \mathcal{L}_\Pi - \mu_\Pi P_2 = 0$ is exact solution of AdS Plateau problem
single segment $\vert$	$\pm$	$\pm y_0 = -sy_1 + cy_2$ $= -y_1 \sin\phi + y_2 \cos\phi$ actually belongs to $\mathcal{R}^{\mp\pm}_{\vert\vert} \subset \mathcal{R}^{\pm}_{\vert}$ $\mathcal{C}_\vert = 1 \pm y_0 - ze^{-i\phi}$	$-$
2 parallel segms $\vert\vert$	$++$ $n = 4k$	$y_0 = y_1 y_2$ actually belongs to $\mathcal{R}_\square \subset \mathcal{R}^{++}_{\vert\vert}$	$-$
square $\square$	$-+-+$ $+-+-$	$y_0 = y_1 y_2$ $-y_0 = y_1 y_2$	$\mu_\square = 0:$ $\mathcal{S}_\square = \mathcal{L}_\square$
rhombus $\diamond$	$-+-+$	$y_0 = y_1 y_2 \cosh\xi + (1 - \frac{1}{2}y^2)\sinh\xi$ $\cosh\xi = \frac{1}{\cos(2\phi)}$, $\sinh\xi = \frac{\sin(2\phi)}{\cos(2\phi)}$	$\mu_\diamond = -\frac{1}{2}\tan(2\phi):$ $\mathcal{S}_\diamond = \mathcal{L}_\diamond - \frac{1}{2}\sinh\xi P_2$ $\sim y_1 y_2 - \frac{1}{2}(1 - y_0^2)\sin(2\phi) - y_0\cos(2\phi)$
2 parallel segms $\vert\vert$	$\mp\pm$ $n = 4k-2$	$\pm y_0 = \mathrm{Im}(ze^{-i\phi})$ $= -sy_1 + cy_2$	$\mu_{\vert\vert} = 0:$ $\mathcal{S}_{\vert\vert} = \mathcal{L}^{-+}_{\vert\vert}$
hexagon	$-+-+-+$	$y_0(1 - \frac{1}{4}y^2) = \frac{1}{4}y_2(3y_1^2 - y_2^2)$	$\mu_{hexa} = y_0 B(y_1, y_2)$ [1], $\mathcal{S}_{hexa} \approx \mathcal{L}_{hexa}$
$Z_n -$ symmetric polygon, n even	alternated	$y_0 Q_n(y^2) = \frac{1}{2^{n/2-1}}\mathrm{Im}(z^{n/2})$	$\mathcal{S}_{Z_n} \approx \mathcal{L}_{Z_n}$, $\mu_{Z_n} = y_0 B_n(y_1, y_2)$ with non $-$ polynomial B_n, see [1]
angle of size 2α $\angle$	$-+$	$y_0\cos\alpha + \sin\alpha = \mathrm{Re}(ze^{-i\theta})$ $= y_1\cos\theta + y_2\sin\theta$	$\mu_\angle = 0:$ $\mathcal{S}_\angle = \mathcal{L}_\angle$ NG eqs are singular in this case, $L_{NG} = 0$
kite	$-+-+$	$y_0\left(1 + y_2\frac{\cos\alpha - \cos\beta}{\sin(\alpha+\beta)}\right) + y_2\frac{\sin\alpha - \sin\beta}{\sin(\alpha+\beta)}$ $+(y_1^2 + y_2^2)\frac{\cos\alpha\cos\beta}{\sin(\alpha+\beta)} - \frac{1}{\sin(\alpha+\beta)}y_2^2$ $-\frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)}$ (y_1, y_2) here are rotated by $\frac{\pi}{4}$ w.r.t.the rhombus and square	$\mu_{kite} = -\frac{1}{2}\cot(\alpha+\beta):$ $\mathcal{S}_{kite} = \mathcal{L}_{kite} + \frac{\cos(\alpha+\beta)}{2\sin(\alpha+\beta)}P_2 \sim$ $\sim y_1^2\cos(\alpha-\beta) - y_2^2\big(2 - \cos(\alpha-\beta)\big)$ $+(y_0^2 - 1)\cos(\alpha+\beta) + 2y_2(\sin\alpha - \sin\beta)$ $+2y_0\sin(\alpha+\beta) + 2y_0 y_2(\cos\alpha - \cos\beta)$
generic skew quadrilateral	$-+-+$	see eqs.(5.55) and (6.65) for two different parametrizations	$\mu_{quadri} = -\frac{t_A t_C - (t_A + t_C)t_B + (2t_A t_C - 1)t_B^2}{2(t_A + t_C)(1 + t_B^2)}$ $\mathcal{S}_{quadri} = \mathcal{L}_{quadri} + \mu_{quadri}P_2$

6.6.3 Solutions to AdS Plateau problem

We are still not in position to describe exact solutions in general situation, even under assumptions (1.2). Still, according to [1], a reasonable approximation can be found within the families of the boundary-ring elements of degree $n/2$:

$$\mathcal{S}_\Pi \approx \mathcal{L}_\Pi - \mu_\Pi(\mathbf{y}) \cdot P_2 \tag{6.83}$$

The optimal choice of the polynomial μ_Π of degree $n/2-2$ can be dictated by two kinds of argument:

• by NG equations, that – according to s.3 – imply that $\mathcal{S}_\Pi$ should be a properly perturbed harmonic function,

• by minimization of regularized area, evaluated as a *height function* on the space of coefficients of μ, as suggested in [13] and [23].

Exact solutions, available at $n=4$ fit into this scheme *exactly*: always belong to the family (6.83), however, unlike in the Z_n-symmetric case considered in [1], the relevant $\mu_\Pi \neq 0$. This means that the third way to specify μ_Π –

• by some algebraic criterium

– still remains to be found: hypothesis $\mathcal{S}_\Pi \overset{?}{\approx} \mathcal{L}_\Pi$ does *not* work for asymmetric Π.

One can easily play with $3d$ plots of above functions to see how nice these approximations are and how strong is dependence on the deviations of $\mu(\mathbf{y})$ from the optimal values. Unfortunately, today such plots can not be adequately represented in a *paper*, even on computer screen, since they necessarily use additional software, allowing to *rotate* $3d$ images. However, after the functions $\mathcal{L}$ are explicitly constructed in this paper, it takes two minutes to write a two-line "program" in MAPLE or Mathematica to make these plots and start investigating them. As explained in [1], it is more informative to plot $r(\vec{y}) = \sqrt{\Big(y_0(y_1,y_2)\Big)^2 + 1 - y_1^2 - y_2^2}$ than $y_0(y_1,y_2)$ itself. As long as $\mu(\mathbf{y})$ is taken to be independent of y_0 the equation $\mathcal{L}+\mu P_2 = 0$ is quadratic for y_0 and can be analytically resolved – this simplifies the computer program even further and makes it working fast on not-very-modern laptops.[9]

If one wants to go beyond *approximate* methods, then for $n>4$ the restriction that μ_Π is a polynomial of degree $n/2-2$, should be lifted. In [1] and s.3 it is shown how one can proceed with *formal series* for $\mu(\mathbf{y})$. It would be most interesting to identify a narrow class of functions, which $\mu_\Pi(\mathbf{y})$ actually belongs to. As shown in s.3, hypergeometric functions can be a better choice than polynomials to address this problem.

7 Appendix. A list of notational agreements

Notations in this paper are somewhat sophisticated, thus it make sense to list them in a separate appendix.

7.1 Polygons and angles

The most difficult are angular variables, associated with our polygons. All of them refer to planar polygons $\bar{\Pi}$, obtained by projection of Π onto the plane (y_1,y_2). Polygon $\bar{\Pi}$ have n sides and n vertices, which we enumerate counterclockwise, assuming that vertex $\#a$ is the intersection of the sides $\#(a-1)$ and $\#a$. In other words, the side a (i.e. external momentum $\mathbf{p}_a$) originates at vertex $\#a$ and ends at vertex $\#(a+1)$. The y_0 variable is either growing or decreasing when we move along this side, this choice is labeled by discrete parameter $\sigma_a = \pm 1$, associated with each side of $\bar{\Pi}$.

The origin of coordinate system in (y_1,y_2) plane is located at the center of the circle, inscribed into $\bar{\Pi}$. In this paper we assume that such circle exists, see discussion around eq.(1.2. This, of course, unjustly restricts the choice of Π, but considerably simplifies the formalism. The general scale is fixed by requiring that the circle radius is unity. Rotational symmetry is not fixed in any universal way, it is done in different ways in different examples, because it is done so in existing literature.

[9] For the sake of convenience we suggest a version of such MAPLE program here:

```
L:= ?? :                                  # for example, for the square L:= y0 − y1 * y2:
mu:=?? :            # function of y1 and y2 with NUMERICAL coefficients should be substituted

P2:= y0^2 + 1 − y1^2 − y2^2:
s:=2          # this parameter can be adjusted to focus on the domain bounded by our polygon

Y:= solve( L + mu*P2, y0 )[1]:  # sometime one needs to change "[1]" for "[2]" to choose appropriate root of quadratic equation
                                # ATTENTION: if mu=0 then there is only one root and "[1]" should be omitted!
plot3d( sqrt(Y^2 + 1 − y1^2 − y2^2), y1:=-s..s, y2:=-s..s, axes=boxed, grid=[100,100] ):
```

The first two lines contain input: explicit expression for $\mathcal{L}_\Pi$ from this paper or [1] and one's favorite parametrization of the trial constant/polynomial/function $\mu(\mathbf{y})$. The last two lines are the plotting program itself. It can be better to substitute trigonometric functions of angles by their rational expressions through tangents of the one-half angle, otherwise MAPLE should be taught trigonometric identities. Before one reaches asymmetric hexagons at $n=6$ one can begin from substituting numbers for μ. For $n \geq 6$ polynomials of y_1 and y_2 of degree $n/2-2$ are a nice starting point. If μ is non-trivial function of y_0 one can need to switch to *pointplot* commands which takes computers more time to work with.

Direction of sides of $\bar{\Pi}$ are defined through directions of *normals* to these sides, which are labeled by angles ϕ_a. This means that direction of the side itself is $\frac{\pi}{2} + \phi_a$. Directions towards the vertices are labeled by the angles θ_a. The difference between θ and ϕ variables is denoted by φ. With above-described convention about comparative enumeration of sides and vertices

$$\theta_a - \varphi_a = \phi_{a-1}, \quad \theta_a + \varphi_a = \phi_a \tag{7.84}$$

Both these formulas contain the same $varphi_a$ – this is a corollary of inscribed-circle condition.

In some examples vertices are also labeled by alphabetically ordered capital letters $1, 2, 3, 4 = A, B, C, D$. Angles of the polygon are denoted $2\alpha_a$, $\alpha_a = \frac{\pi}{2} - \varphi_a$ is one *half* of the polygon angle.

7.2 Boundary rings and exact solutions

"Linear in y_0" means that the expression has the form $Ay_0 + B$ with B not necessarily vanishing. This is the usual form of our canonical element $\mathcal{L}_\Pi = y_0 Q_\Pi(y_1, y_2) - \mathcal{K}_\Pi(y_1, y_2)$.

Multiplicative character is a number-valued homomorphism of the ring multiplication. When we consider a union, $\Pi = \Pi_1 \cup \Pi_2$, the boundary rings are multiplied and so do characters: a family of functions $C_\Pi(y_0, y_1, y_2)$ is a multiplicative character if $C_\Pi = C_{Pi_1} C_{\Pi_2}$. Examples of multiplicative characters are "obvious" elements (2.6) of the polygon boundary rings and also the complex-valued $\mathcal{C}_\Pi$ from s.6.2.

Calligraphic letters denote elements of the boundary rings, as well as the rings themselves. However there are exceptions, not *all* elements of the ring are denoted by calligraphic letters and some objects, though denoted by calligraphic letters, do not belong to the ring. Among the elements of the ring $\mathcal{R}_\Pi$ are: complex characters $\mathcal{C}_\Pi$, canonical y_0-linear elements $\mathcal{L}_\Pi$, most of solutions $\mathcal{S}_\Pi$ to AdS Plateau problem mentioned in this paper. However, real-valued characters (2.6) are also elements $\mathcal{R}_\Pi$, still they are denoted by ordinary capital letters P. This is because $\mathcal{P}_\Pi$ was used in [1] and in s.2.5 to denote a "nice" element of $\mathcal{R}_\Pi$ – a notion that we still did not manage to extend beyond Z_n-symmetric case in the present paper. For Z_n -symmetric Π this $\mathcal{P}_\Pi = \mathcal{L}_\Pi$ and simultaneously $\mathcal{S}_\Pi \approx \mathcal{P}_\Pi$, but this in general $\mathcal{S}_\Pi \neq \mathcal{L}_\Pi$. The difference is measured by μ_Π, which is constant for exact solutions considered in this paper, and this constant is non-vanishing in asymmetric situations (starting from rhombus). In general, for $n > 4$, μ_Π is not a constant and, perhaps, not even a polynomial, this means that in general $\mathcal{S}_\Pi$ is not quite an element of the polynomial boundary ring $\mathcal{R}_\Pi$, it rather belongs to some completion $\overline{\mathcal{R}_\Pi}$, which can hopefully be made smaller than just the formal series made from elements of $\mathcal{R}_\Pi$. The prototype of μ_Π is called $\mathcal{B}$ in s.2.5, despite denoted by calligraphic letter, it is *not* and element of the boundary ring or of its completion: $P_2\mathcal{B}$ is. The same is true about $\mathcal{K}_\Pi$: it does not belong to the ring, $\mathcal{L}_\Pi = y_0 Q_\Pi - \mathcal{K}_\Pi$ does.

Acknowledgements

We appreciate collaboration and discussions with A.Mironov on the main topics of his paper. H.Itoyama acknowledges the hospitality of ITEP during his visit to Moscow at the beginning of this work. A.Morozov is indebted for hospitality to Osaka City University and for support of JSPS. The work of H.I. is partly supported by Grant-in-Aid for Scientific Research 18540285 from the Ministry of Education, Science and Culture, Japan and the XXI Century COE program "Constitution of wide-angle mathematical basis focused on knots", the work of A.M. is partly supported by Russian Federal Nuclear Energy Agency, by the joint grant 06-01-92059-CE, by NWO project 047.011.2004.026, by INTAS grant 05-1000008-7865, by ANR-05-BLAN-0029-01 project and by the Russian President's Grant of Support for the Scientific Schools NSh-8004.2006.2, and by RFBR grant 07-02-00645.

References

[1] H.Itoyama, A.Mironov and A.Morozov, *Boundary Ring or a Way to Construct Approximate NG Solutions with Polygon Boundary Conditions. I. Z_n-Symmetric Configurations*, arXiv:0712.0159

[2] A.Polyakov, *Quantum Geometry of Bosonic Strings*, Phys.Lett. **B103** (1981) 207-210;
A.Polyakov, *Gauge Fields and Strings*, 1987;
A.Polyakov, *String Theory and Quark Confinement*, Nucl.Phys.Proc.Suppl. 68 (1998) 1-8, hep-th/9711002

[3] J.Maldacena, *The Large N Limit os Superconformal Field Theories and Supergravity*, Adv.Theor.Math.-Phys. **2** (1998) 231-252; Int.J.Theor.Phys. **38** (1999) 1113-1133; hep-th/9711200;
S.Gubser, I.Klebanov and A.Polyakov, *Gauge Theory Correlators from Non-Critical String Theory*, Phys.Lett. B428 (1998) 105-114; hep-th/9802109;
E.Witten, *Anti de Sitter Space and Holography*, Adv.Theor.Math.Phys. **2** (1998) 253-291, hep-th/9802150

[4] L.Alday and J.Maldacena, *Gluon Scattering Amplitudes at Strong Coupling*, arXiv:0705.0303

[5] S.Abel, S.Forste and V.Khose, *Scattering Amplitudes in Strongly Coupled $N = 4$ SYM from Semiclassical Strings in AdS*, arXiv:0705.2113

[6] E.Buchbinder, *Infrared Limit of Gluon Amplitudes at Strong Coupling*, arXiv:0706.2015

[7] J.Drummond, G.Korchemsky and E.Sokatchev, *Conformal properties of four-gluon planar amplitudes and Wilson loops*, arXiv:0707.0243

[8] A.Brandhuber, P.Heslop and G.Travaglini, *MHV Aplitudes in $N = 4$ Super Yang-Mills and Wilson Loops*, arXiv:0707.1153

[9] F.Cachazo, M.Spradlin and A.Volovich, *Four-Loop Collinear Anomalous Dimension in $N = 4$ Yang-MillsTheory*, arXiv:0707.1903

[10] M.Kruczenski, R.Roiban, A.Tirziu and A.Tseytlin, *Strong-Coupling Expansion of Cusp Anomaly and Gluon Amplitudes from Quantum Open Strings in $AdS_5 \times S^5$*, arXiv:0707.4254

[11] Z.Komargodsky and S.Razamat, *Planar Quark Scattering at Strong Coupling and Universality*, arXiv:0707.4367

[12] A.Jevicki, C.Kalousios, M.Spradlin and A.Volovich, *Dressing the Giant Gluon*, arXiv:0708.0818

[13] A.Mironov, A.Morozov and T.N.Tomaras, *On n-point Amplitudes in N=4 SYM*, JHEP **0711** (2007) 021, arXiv:0708.1625

[14] H.Kawai and T.Suyama, *Some Implications of Perturbative Approach to AdS/CFT Correspondence*, arXiv:0708.2463

[15] S.G.Naculich and H.J.Schnitzer, *Regge behavior of gluon scattering amplitudes in N=4 SYM theory*, arXiv:0708.3069

[16] R.Roiban and A.A.Tseytlin, *Strong-coupling expansion of cusp anomaly from quantum superstring*, arXiv:0709.0681

[17] J.M.Drummond, J.Henn, G.P.Korchemsky and E.Sokatchev, *On planar gluon amplitudes/Wilson loops duality*, arXiv:0709.2368

[18] D.Nguyen, M.Spradlin and A.Volovich, *New Dual Conformally Invariant Off-Shell Integrals*, arXiv:0709.4665

[19] J.McGreevy and A.Sever, *Quark scattering amplitudes at strong coupling*, arXiv:0710.0393

[20] L.Alday and J.Maldacena, *Comments on Operators with Large Spin*, arXiv:0708.0672; *Comments on gluon scattering amplitudes via AdS/CFT*, arXiv:0710.1060

[21] S.Ryang, *Conformal SO(2,4) Transformations of the One-Cusp Wilson Loop Surface*, arXiv:0710.1673

[22] D.Astefanesei, S.Dobashi, K.Ito and H.S.Nastase, *Comments on gluon 6-point scattering amplitudes in N=4 SYM at strong coupling*, arXiv:0710.1684

[23] A.Mironov, A.Morozov and T.Tomaras, *Some properties of the Alday-Maldacena minimum*, to appear in Phys.Lett.**B**, arXiv:0711.0192,

[24] A.Popolitov, *On coincidence of Alday-Maldacena-regularized σ-model and Nambu-Goto areas of minimal surfaces*, Pis'ma v ZhETF **86** #9 (2007) 643-645, arXiv:0710.2073

[25] Gang Yang, *Comment on the Alday-Maldacena solution in calculating scattering amplitude via AdS/CFT*, arXiv:0711.2828

[26] K.Ito, H.S.Nastase and K.Iwasaki, *Gluon scattering in $\mathcal{N} = 4$ Super Yang-Mills at finite temperature*, arXiv:0711.3532

[27] A.Jevicki, K.Jin, C.Kalousios and A.Volovich, *Generating AdS String Solutions*, arXiv:0712.1193

[28] J.M.Drummond, J.Henn, G.P.Korchemsky and E.Sokatchev, *Conformal Ward identities for Wilson loops and a test of the duality with gluon amplitudes*, arXiv:0712.1223

[29] Z.Bern, L.Dixon and V.Smirnov, *Iteration of Planar Amplitudes in Maximally Supersymmetric Yang-Mills Theory at Three Loops and Beyond*, Phys.Rev. **D72** (2005) 085001, hep-th/0505205

[30] The problem is named after Joseph Antoine Ferdinand Plateau, a nineteenth-century, blind, Belgian physicist, who "observed" a handful of simple patterns that seemed to completely describe the geometry of how soap bubbles fit together. For a little more details see, for example,
http://scidiv.bcc.ctc.edu/math/mathematics/Plateau.html

[31] R.Kallosh and A.Tseytlin, *Simplifying Superstring Action on $AdS_5 \times S^5$*, JHEP **9810** (1998) 016, hep-th/9808088

[32] M.Kruczenski, *A Note on Twist Two Operators in $N = 4$ SYM and Wilson Loops in Minkowski Signature*, JHEP **0212** (2002) 024, hep-th/0212115

[33] See discussion of Schwarz reflection principle at the bottom of page 16 of English edition or at page 28 of Russian edition in P.Hoffman and H.Karcher, *Complete Embedded Minimal Surfaces of Finite Total Curvature*, in *Encyclopaedia of Math.Science*, **90**, *Geometry V. Minimal Surfaces*, ed.R.Osserman, Springer

[34] A.Morozov and M.Serbyn, *Non-Linear Algebra and Bogolubov's Recursion*, to appear in Theor.Math.Phys., hep-th/0703258

[35] V.Dolotin and A.Morozov, *The Universal Mandelbrot Set, Beginning of the Story*, World Scientific, 2006; hep-th/0501235; hep-th/0701234;
Andrey Morozov, *Universal Mandelbrot Set as a Model of Phase Transition Theory*, Pis'ma v ZhETF **86** #11 (2007) 856-859, arXiv:0710.2315;
Sh.Shakirov, *Higher discriminants of polynomials*, Theor.Math.Phys. **153(2)** (2007) 1477-1486; math/0609524

[36] V.Dolotin and A.Morozov, *Introduction to non-linear Algebra*, World Scientific, 2007; hep-th/0609022;
A.Anokhina et al, *to appear*

[37] S.Lang, *Algebra*, Addison-Wesley Seires in Mathematics, 1965
B.L.Van der Varden, *Algebra*, I, II, Springer-Verlag, 1967, 1971

[38] A.Cayley, *On the Theory of Linear Transformations*, Camb.Math.J. **4** (1845) 193-209;
see [40] for a recent show-up of Cayley's $2 \times 2 \times 2$ hyperdeterminant in string-theory literature, where it appears in the role of the $SL(2)^3$ invariant

[39] A.Levin and A.Morozov, *On the foundations of the random lattice approach to quantum gravity*, Phys.Lett. **B243** (1990) 207-214

[40] A.Miyake and M.Wadati, *Multiparticle Entaglement and Hyperdeterminants*, quant-ph/02121146;
V.Coffman, J.Kundu and W.Wooters, *Distributed Entaglement*, Phys.Rev. **A61** (2000) 52306, quant-ph/9907047;
M.J.Duff, *String Triality, Black Hole Entropy and Cayley's Hyperdeterminant*, hep-th/0601134; *Hidden Symmetries of the Nambu-Goto Action*, hep-th/0602160;
R.Kallosh and A.Linde, *Strings, Black Holes and Quantum Information*, hep-th/0602061